Week 9: Integration over General Regions

y-Simple Regions (M&T p. 283)

A region $R \subset \mathbb{R}^2$ is <u>y-simple</u> if there are continuous functions ϕ_1 and $\phi_2 : [a,b] \to \mathbb{R}$ such that:

$$R = \{(x,y) : a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$$

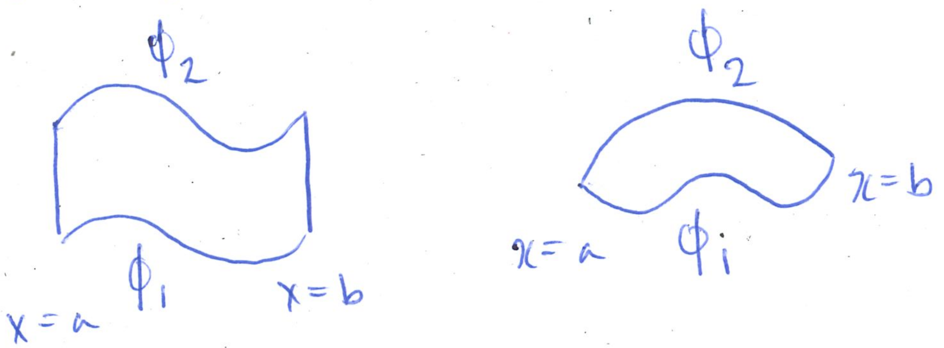
"simple ≈ banded"

x-Simple Regions (M&T p. 283)

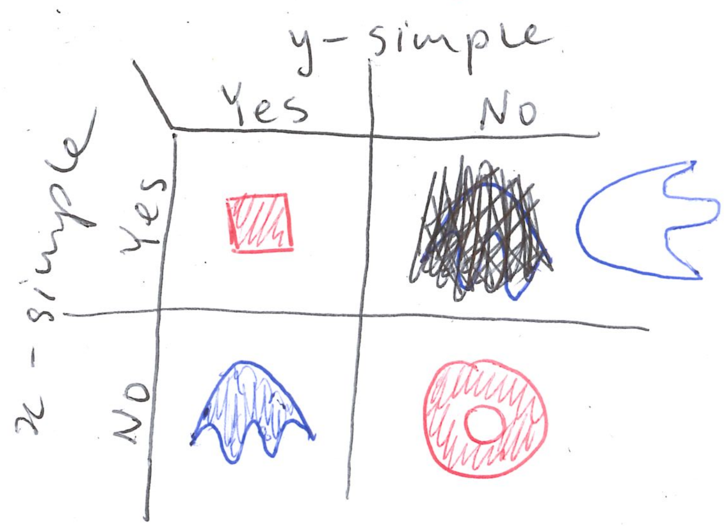
A region $R \subset \mathbb{R}^2$ is <u>x-simple</u> if there are continuous functions ψ_1 and $\psi_2 : [c,d] \to \mathbb{R}$ such that:

$$R = \{(x,y) : c \le y \le d : \psi_1(y) \le x \le \psi_2(y)\}$$

y-simple regions



This terminology "y-simple" and "x-simple" is NOT standard but our book uses it.

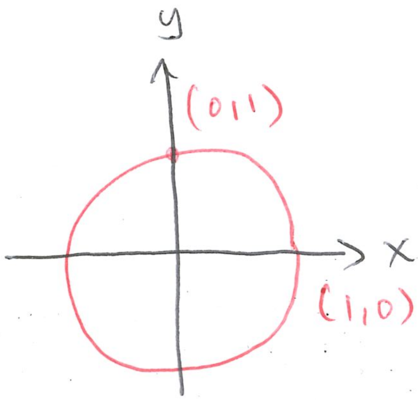


Simple Regions (M&T p. 283)

A region $R \subset \mathbb{R}^2$ is simple if it is x-simple and y-simple.

Example: The Disk is Simple

Show that $R = \{(x, y) : x^2 + y^2 \leq 1\}$ is simple.



Show R is y-simple.

$$R = \left\{ (x,y) : \begin{array}{l} -1 \leq x \leq 1 \\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \end{array} \right\}$$

This comes from

$$x^2 + y^2 = 1 \Leftrightarrow y = \pm\sqrt{1-x^2}$$

Show R is x-simple

$$R = \left\{ (x,y) : \begin{array}{l} -1 \leq y \leq 1 \\ -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \end{array} \right\}$$

Double Integrals for y-Simple Regions (M&T p. 286)

If R is y-simple we have:

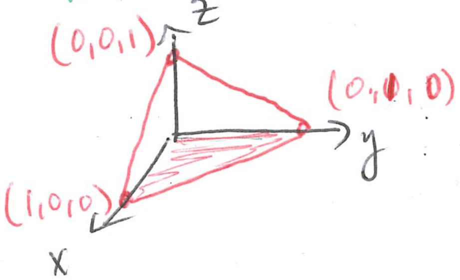
$$\iint_R f(x,y)\, dA = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy \right] dx$$

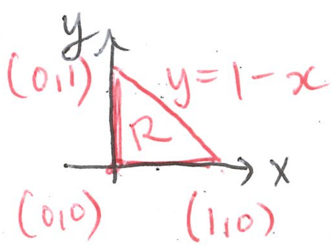
we need variable in bound and the variable in outer integral match.

Example: The Volume of a Pyramid

Find the volume of the region bounded described by $0 \le x, y, z$ and $z \le 1 - x - y$.

Sketch the region





$$0 = 1 - x - y$$

$$\Leftrightarrow x + y = 1$$

$$\Leftrightarrow y = 1 - x$$

Setup the integral

$$V = \iint_R 1 - x - y \, dA$$

$$= \int_0^1 \int_0^{1-x} 1 - x - y \, dy \, dx$$

$$= \int_0^1 \left[y - xy - \frac{1}{2} y^2 \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 (1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \, dx$$

$$= \int_0^1 1 - x - x + x^2 - \frac{1}{2} + x - \frac{1}{2} x^2 \, dx$$

$$= \int_0^1 \frac{1}{2} - x + \frac{1}{2} x^2 \, dx = \left[\frac{1}{2} x - \frac{1}{2} x^2 + \frac{1}{6} x^3 \right]_0^1$$

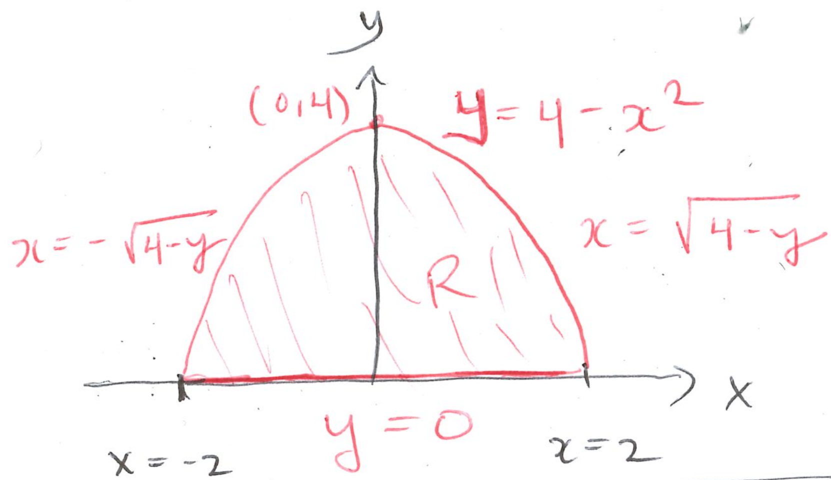
$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

Example: Sketch the Region

Sketch the region of integration for the following integral.

$$\int_{-2}^{2}\int_{0}^{4-x^2} f(x,y)\,dydx$$

y-simple



This region is also x-simple!

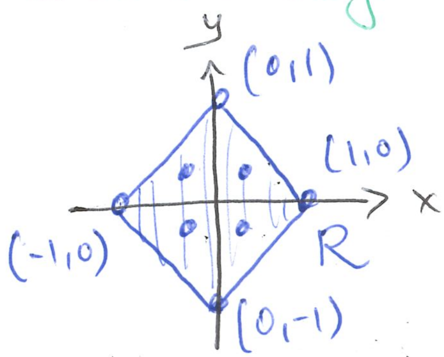
$$\int_{-2}^{2}\int_{0}^{4-x^2} f(x,y)\,dydx = \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x,y)\,dx\,dy$$

$f(x,y) = x^2+y^2$ satisfies $f(x,y)=f(-x,y)=f(x,-y)=f(-x,-y)$

Example: A Symmetry Argument

Consider the region R with vertices $(0, \pm 1)$ and $(\pm 1, 0)$. Evaluate the integral $\iint_R (x^2+y^2)\, dydx.$ $= \frac{4}{6} = \frac{2}{3}$

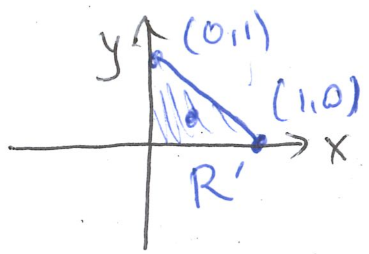
#Sketch region



By symmetry:

$$\iint_R (x^2+y^2)\,dydx = 4\iint_{R'} (x^2+y^2)\,dydx$$

where R' has vertices $(0,0)$, $(1,0)$, $(0,1)$.



$$\iint_{R'} x^2+y^2\,dydx = \int_0^1\int_0^{1-x} x^2+y^2\,dydx$$

$$= \int_0^1 \left[x^2y+\frac{1}{3}y^3\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 x^2(1-x)+\frac{1}{3}(1-x)^3\,dx = \frac{1}{6}$$

! This has $y=0$ as a bounding function.

Also, by symmetry: $\iint_R x^2\,dA = \iint_R y^2\,dA$

Activity: Generalize y-simple to x-simple

If R is x-simple then $R = \{ a \leq y \leq b : \psi_1(y) \leq x \leq \psi_2 \}$ and we have:

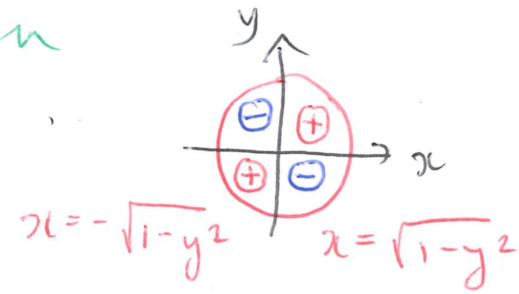
$$\iint_R f(x,y)\, dA = \int_a^b \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx \right] dy$$

These must match.

Example: Setup an Integral

Setup the integral $\iint_R xy\, dxdy$ where $R = \{(x,y) : x^2 + y^2 = 1\}$.

Setup as an x-simple region



$$\iint_R xy\, dxdy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xy\, dxdy$$

Follow-up Question: Can you guess the value of the integral? $\frac{1}{2}$? $-\frac{1}{2}$? π? 0?

$\iint_R xy\, dxdy = 0$ because $f(-x,y) = -f(x,y)$ "odd function"

and the region R is symmetric under $(x,y) \mapsto (-x,y)$

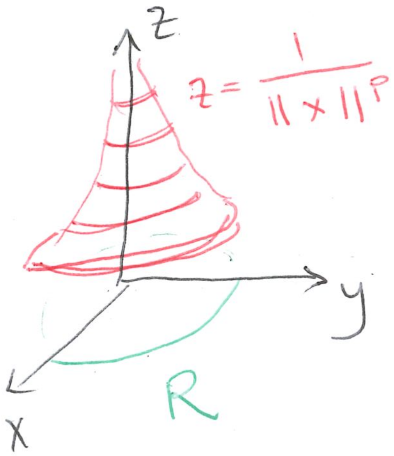
Example: An Improper Integral

For what values $p > 0$ does the following integral converge?

$$V(p) = \iint_R \frac{1}{\|(x,y)\|^p}\, dA$$

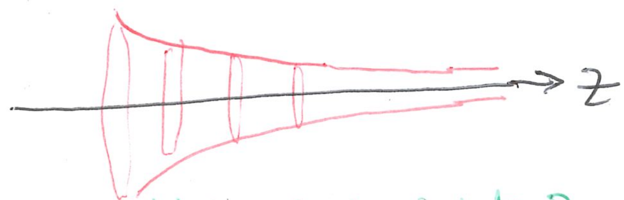
where $R = \{(x,y) : 0 < x^2 + y^2 \leq 1\}$.

The integrand is unbounded so the integral is improper.



Integrate along the z-axis using Cavalier's Principle.

Why? Improper and (currently) circles are hard.



Calculate radius of disk at height z.

$$z = \frac{1}{\|(x,y)\|^p} \Leftrightarrow z^{-\frac{1}{p}} = \|(x,y)\| = r.$$

Calculate V(p)

$$V(p) = \int_1^\infty \pi r^2\, dz = \int_1^\infty \pi z^{-2\frac{1}{p}}\, dz = \pi \int_1^\infty z^{-2\frac{1}{p}}\, dz$$

$$= \pi \lim_{t\to\infty} \int_1^t z^{-2\frac{1}{p}}\, dz = \pi \lim_{t\to\infty} \left[\frac{1}{-2p+1} z^{-2\frac{1}{p}+1} \right]_1^t$$

We need $-2\frac{1}{p}+1 < 0$ for this limit to exist.

Equivalently, $V(p)$ converges for $p < 2$.

(Details on back of p. 107.)

$$-2\frac{1}{p}+1<0 \iff -2\frac{1}{p}<-1$$
$$\iff 2\frac{1}{p}>1$$
$$\iff \frac{1}{p}>\frac{1}{2}$$
$$\iff p<2$$

Changing the Order of Integration (M&T p. 290)

If R is x-simple and y-simple then we have:

$$\iint_R f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dydx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dxdy$$

This is Fubini's Theorem for simple regions.

(!) Both sets of bounds can change when we change the order of integration.

For rectangles:

$$\int_a^b \int_c^d = \int_c^d \int_a^b$$

Try for 3 min!

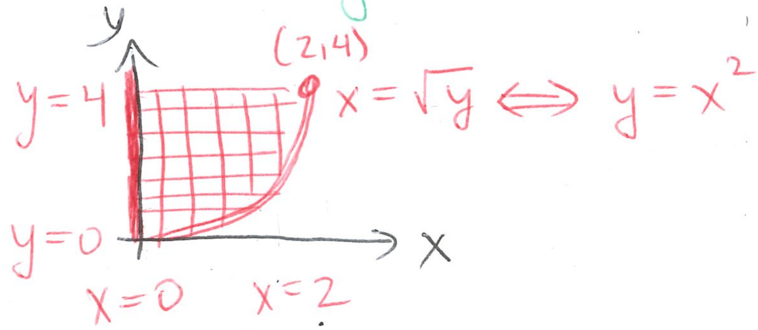
Example: Sketch and Change #1

Sketch the region of integration and change the the order of integration.

$$\int_0^4 \int_0^{\sqrt{y}} f(x,y)\, dx dy$$

x-simple

Sketch the region

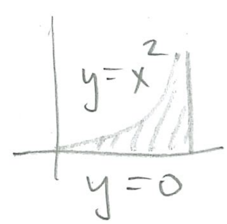


Setup the alternate order.

$$\int_0^2 \int_{x^2}^4 f(x,y)\, dy\, dx$$

why not $\int_0^2 \int_0^{x^2} f(x,y)\, dy dx$?

This is the region below the parabola.



The original region was above the parabola.

Try for 3 min

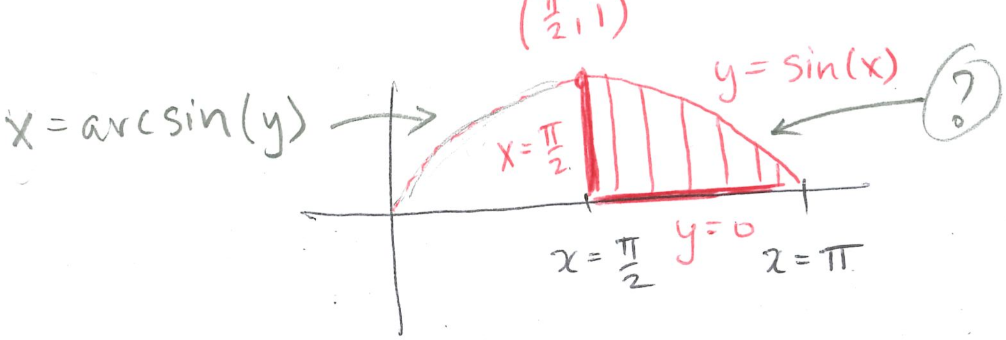
Example: Sketch and Change #2

Sketch the region of integration and change the the order of integration.

$$\int_{\pi/2}^{\pi}\int_{0}^{\sin(x)} f(x,y)\,dydx$$

y-simple

Sketch the region



Setup integral in alternate order

$$\int_0^1 \int_{\frac{\pi}{2}}^{\quad} f(x,y)\,dx\,dy$$

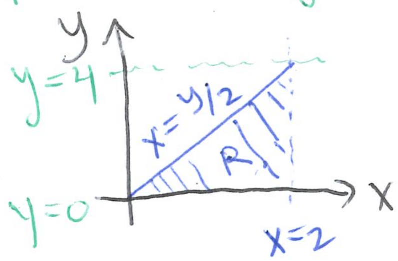
$-\arcsin(y) + \pi$? $\arcsin(-y) + \pi$?

Example: From Impossible to Possible

Calculate $\int_0^4 \int_{y/2}^2 e^{x^2}\, dx dy$.

It is impossible to integrate $\int e^{x^2} dx$ using simple functions.

Sketch region of integration # Change order



$x = \frac{y}{2} \iff y = 2x$

$$\int_0^2 \int_0^{2x} e^{x^2} \, dy dx$$

$$= \int_0^2 \left[y e^{x^2} \right]_{y=0}^{y=2x} dx$$

$$= \int_0^2 2x e^{x^2} dx$$

$u = x^2$ $\quad u(0) = 0^2 = 0$

$du = 2x\,dx$ $\quad u(2) = 2^2 = 4$

$$= \int_0^4 e^u du = \left[e^u \right]_0^4$$

$$= e^4 - e^0 = e^4 - 1$$

whoa!

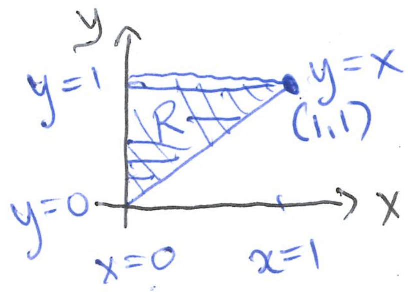
Q: Is there a specific property of a function that causes it to not have an elementary anti-derivative?

Differential Galois Theory

Example: Another Impossible Integral

Calculate $\int_0^1 \int_0^y \sin(x^2)\ dxdy$.

Sketch the region # Change order of integration



$$\int_0^1 \int_x^1 \sin(x^2)\, dy dx$$

$$= \int_0^1 \Big[y\sin(x^2) \Big]_x^1 dx$$

$$= \int_0^1 \sin(x^2) - \underbrace{x\sin(x^2)}_{\text{possible}}\, dx$$

$$= \int_0^1 (1-x)\sin(x^2)\, dx$$

It is not 100% clear that this is (im)possible.

 Version: 1.0

Mean-Value Theorem for Integrals (M&T p. 292)

If $f : R \to \mathbb{R}$ is continuous and R is simple then there is some point (x_0, y_0) in R such that:

$$\iint_R f(x,y)\, dA = f(x_0,y_0)\,\text{Area}(R)$$

Big Idea: Bound the integral of f by its maximum and minimum time the area and then use continuity.

R is simple and so it is closed and bounded.

Thus, $f(x,y)$ has an absolute max and absolute min on R.

Suppose $\boxed{m \leq f(x,y) \leq M}$ are the absolute min and absolute Max.

This gives

$$m\,\text{Area}(R) \leq \iint_R f(x,y)\,dA \leq M\,\text{Area}(R)$$

$$\Leftrightarrow \quad m \leq \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA \leq M.$$

By the intermediate value theorem, there is $(x_0\ y_0)$ with:

$$\boxed{f(x_0,y_0)} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$$

$$\Leftrightarrow \quad \text{Area}(R)\, f(x_0,y_0) = \iint_R f(x,y)\,dA.$$