Week 10: The Triple Integral

The Triple Integral as a Riemann Sum (M&T p. 294)

For a bounded function $f : \mathbb{R}^3 \to \mathbb{R}$ we define:

$$\iiint_R f\, dV = \lim_{N\to\infty} \sum_{i,j,k=0}^{n-1} f(\mathbf{c}_{ijk})\Delta x \Delta y \Delta z.$$

If this limit exists and is independent of $\mathbf{c}_{ijk}$ then f is integrable.

Fubini for Rectangular Prisms (M&T p. 295)

If $R = [a,b]\times[c,d]\times[p,q]$ we have:

$$\iiint_R f dV = \int_a^b \int_c^d \int_p^q f dzdydx = \int_p^q \int_c^d \int_a^b f\, dxdydz = \int_p^q \int_a^b \int_c^d f\, dydxdz = \cdots$$

All six orders of integration will agree.

Example: Pick an Order and Integrate

Calculate $\iiint_B x^2\, dV$ where $B = [0,1]^3$.

Pick an order of integration

$$\iiint_B x^2 dV = \int_0^1\int_0^1\int_0^1 x^2 dx\, dy dz = \int_0^1\int_0^1 \left[\frac{1}{3}x^3\right]_{x=0}^{x=1} dydz$$

$$= \int_0^1\int_0^1 \frac{1}{3} dydz = \frac{1}{3}(1-0)(1-0) = \frac{1}{3}$$

We have $\iiint_B x^2 dV = \iiint_B y^2 dV = \iiint_B z^2 dV = \frac{1}{3}$

Follow-Up Question: What is the value of integral $\iiint_B x^2 + y^2 + z^2\, dV$? $= 3\left(\frac{1}{3}\right) = 1$

$dV_n = \Delta x_1 \Delta x_2 \cdots \Delta x_n$

Theorem: Averages

Suppose $R \subseteq \mathbb{R}^n$. Let $f : R \to \mathbb{R}$ be an integrable function. The average of f on R is:

$$A = \frac{1}{\mathrm{Vol}_n(R)} \int_R f \, dV_n$$

Vol_n is the n-dimensional volume of R and dV_n is the n-dimensional volume element.
Warning: We wrote the single integral sign to stand for n-integrations.

Example: An Intriguing n-Dimensional Phenomena

Let $||\mathbf{x}||^2$ be the distance squared function on $\mathbb{R}^n$. What is the average of $||\mathbf{x}||^2$ on $[0,1]^n$?

← The cube is a "small" shape.

$$A = \frac{1}{\mathrm{vol}_n(B)} \int_B ||x||^2 \, dV_n = \frac{1}{(1-0)^n} \int_B ||x||^2 \, dV_n$$

$$= \int_B ||x||^2 \, dV_n = \int_B x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 \, dV_n$$

$$= \frac{1}{3} n$$ ← This is VERY surprising. The average distance is HUGE!

What is the largest distance?

$$d(0 \ldots 0,\ 1 \ldots 1) = \sqrt{(1-0)^2 + \cdots + (1-0)^2} = \sqrt{n}$$

Elementary Regions (M&T p. 297)

The notion of x-simple and y-simple generalizes ~~the~~ to $\mathbb{R}^n$ as elementary. A region is z-elementary if it is bounded on two sides by graphs of functions.

$$R = \{(\mathbf{x}, z) : f_1(\mathbf{x}) \leq z \leq f_2(\mathbf{x})\} \subset \mathbb{R}^{n+1} = \underbrace{\mathbb{R}^n}_{x} \times \underbrace{\mathbb{R}^1}_{z}$$

A region is a symmetric elementary region if it is elementary in all directions.
Warning: The word "symmetric" is being used in a special way here.

Example: Sphere

Show that the sphere $R = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ is a symmetric elementary region.

Bounded by graphs in x-direction
(Alternatively: x-elementary.)

$$x^2+y^2+z^2=1 \iff x = \pm\sqrt{1-y^2-z^2}$$

We get: $-\sqrt{1-y^2-z^2} \leq x \leq \sqrt{1-y^2-z^2}$

y-elementary

$$-\sqrt{1-x^2-z^2} \leq y \leq \sqrt{1-x^2-z^2}$$

z-elementary

$$-\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}$$

Thus, R is bounded by graphs of functions in all three direction.

Therefore, R is a symmetric elementary region.

Q: Is $\mathbb{R}^3$ a symmetric elementery region?

No! It is not bounded by graphs of functions.

we have that z is unbounded on $\mathbb{R}^3$.

So — $\mathbb{R}^3$ is not z-elementary.

Example: A Triple Integral

Compute the triple integral $\iiint\limits_R xyz^2\,dV$ where: $R = [0,1] \times [-1,2] \times [0,3]$.

y z

Pick an order of integration.

$$\int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \,dx\,dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \left[\frac{1}{2}x^2 y z^2\right]_{x=0}^{x=1} dy\,dz$$ # Integrate x.

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} y z^2 \,dy\,dz$$ # Integrate y

$$= \int_0^3 \left[\frac{1}{4} y^2 z^2\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3}{4} z^2\,dz = \left[\frac{1}{4}z^3\right]_{z=0}^{z=3}$$ # Integrate z.

$$= \frac{27}{4}$$

Setting up triple integrals is tricky.
Humans are 2D animals, not 3D.

Example: A General Region

Compute the integral $\iiint\limits_R z\, dV$ where $R = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq x, 0 \leq z \leq 12xy\}$.

" Most explicit bounds should be outermost in the triple integral."

$$\iiint\limits_R z\,dV = \int_0^1 \cdots\cdots dx$$

$0 \leq x \leq 1$ is our most explicit bound

$$= \int_0^1 \int_0^x \cdots\cdots dy\,dx$$

$$= \int_0^1 \int_0^x \int_0^{12xy} z\,dz\,dy\,dx$$

Double check that all variables appearing in bounds will get integrated out eventually.

$= 4$

Exercise: Calculate this!

Example: A Triple Integral

Calculate $\iiint_R e^{z/y}\, dV$ where $R = \{(x,y,z) : 0 \le y \le 1,\ \underbrace{y \le x \le 1}_{\text{Huh?}},\ 0 \le z \le xy\}$.

Setup the integral
$0 \le y \le 1$ is very explicit $\Rightarrow$ outermost

$$\iiint_R e^{z/y}\, dV = \int_0^1 \text{- - - - -}\; dy$$

we have bounds for y
$\Rightarrow$ $y \le x \le 1$ is an ~~x~~ x-bound

$$= \int_0^1 \int_{y}^1 \text{- - -}\; dx\, dy$$

$0 \le z \le xy$ must be a z-bound.

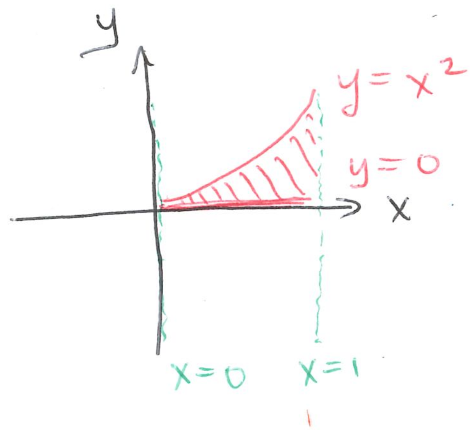
$$= \int_0^1 \int_y^1 \int_0^{xy} e^{z/y}\, dz\, dx\, dy$$

Everything gets integrated out by the end.

Final answer const.

Example: Projection on to a Plane

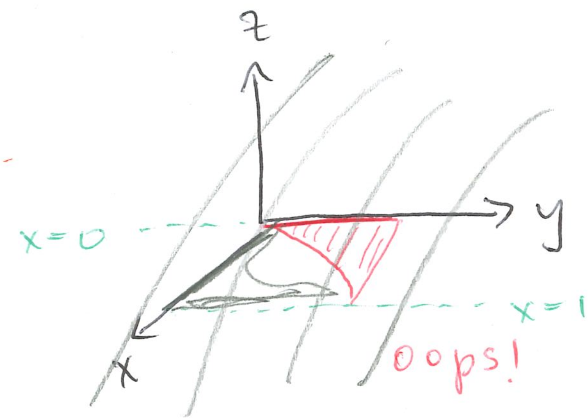
Sketch the region of integration of $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dzdydx$.
Sketch its projection on to the plane $z = 0$. [Explore on CalcPlot3D]



On the plane $z = 0$ we have the bounds:

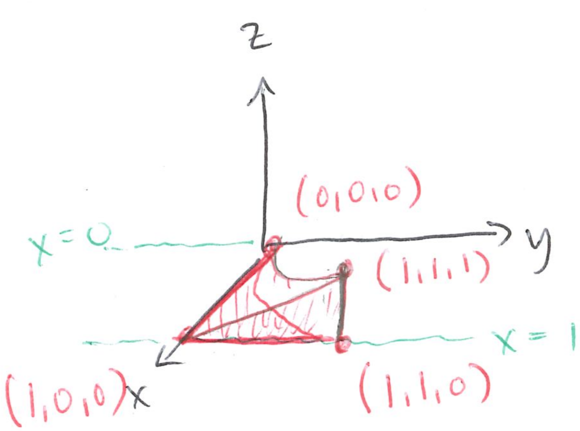
$0 \leq x \leq 1$

$0 \leq y \leq x^2$



On $\mathbb{R}^3$ we have the bound

$0 \leq z \leq y$



For plots in CalcPlot3D:

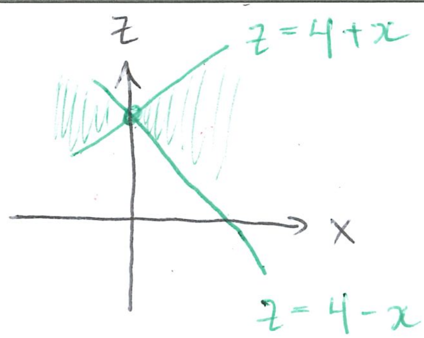
→ Project to $z=0$ plane
Get "2D Top" and "2D Bottom"

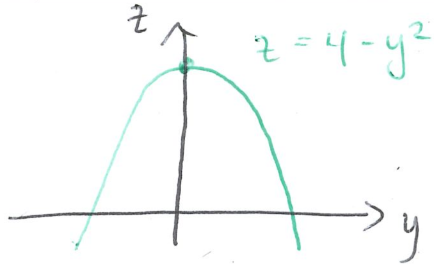
→ Use z-elementary to get "3D Top" "3D Bottom"

Example: Projection on to a Plane

Sketch the region of integration bounded by $z = 4 - x$, $z = 4 + x$, $z = 4 - y^2$. Sketch its projection on to the plane $x = 0$.

$0 \leq z \leq 4 - x$
$0 \leq z \leq 4 + x$



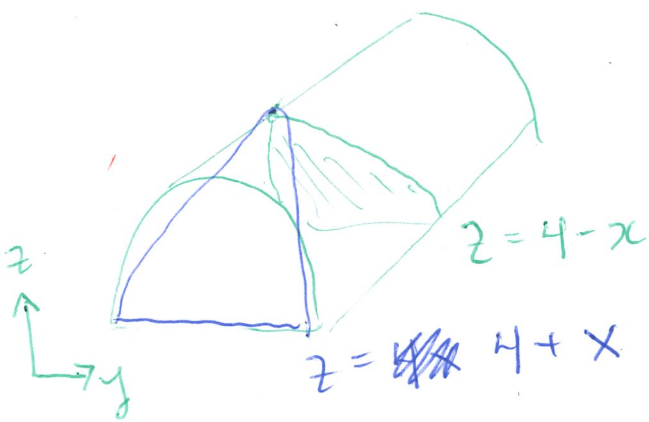


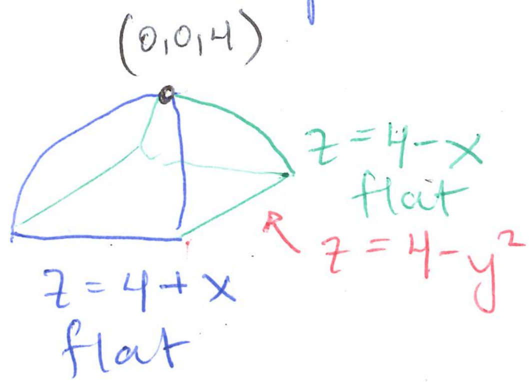
x = 0 plane

When x = 0 we get z = 4

We have $(x, y, z) = (0, 0, 4)$ is in the region.

Q: Should we switch to the complement?



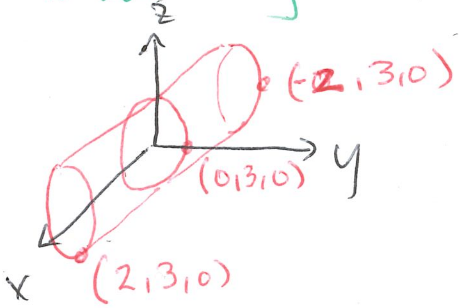


Exercise: Sketch in CalcPlot3D.

Example: All Six Orders

Write all six orders of integration for the region bounded by $y^2 + z^2 = 9$, $x = -2$, $x = 2$.

Quickly Sketch the Region



Start with explicit bounds

$$\int_{-2}^{2} \cdots\cdots dx$$

Introduce y-bounds

$$\int_{-2}^{2}\int_{-3}^{3} \cdots\cdots dy\,dx$$

$$y^2+z^2=9$$
$$\iff z = \pm\sqrt{9-y^2}$$

$$\int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dy\,dx$$

write the order dydzdx

$$\int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dz\,dx$$

write the order dzdxdy

$$\int_{-3}^{3} \cdots\cdots dy = \int_{-3}^{3}\int_{-2}^{2} \cdots\cdots dx\,dy = \int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f\,dz\,dx\,dy$$

write the order dxdzdy (Fubini is nice here!)

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-2}^{2} f\,dx\,dz\,dy$$

See back of p.125

write the order $dx\,dy\,dz$

$$\int_{-3}^{3} \cdots\cdots dz$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \cdots\cdots dy\,dz$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{-2}^{2} f \, dx\,dy\,dz$$

Fubini on rectangles.

write the order $dy\,dx\,dz$

$$\int_{-3}^{3} \int_{-2}^{2} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f \, dy\,dx\,dz$$

Essentially the cylinder is a product. So, Fubini applied to lots of rectangles.

If your order of integration produces variables in the outermost bounds, something went wrong.

(for questions like $\iiint_R f \, dV$.)

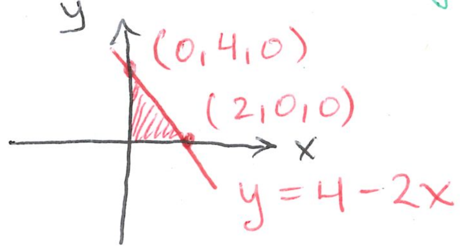
Volume (M&T p. 297)

The volume of $R \subset \mathbb{R}^3$ is $V = \iiint_R 1\, dV$.

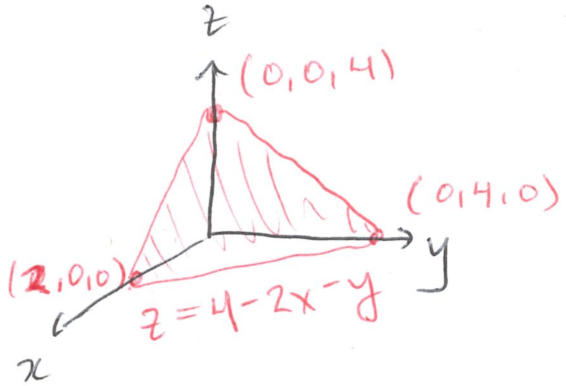
Example: Volume of a Tetrahedron

Find the volume of the tetrahedron bounded by $0 \le x, y, z$ and $z \le 4 - 2x - y$. ← height of tetrahedron

Sketch the region in z=0 plane



$$0 = 4 - 2x - y \iff 2x + y = 4$$



Setup the integral

$$V = \iiint_R 1\, dV$$

$$= \int_0^2 \cdots\cdots dx \quad \# \; 0 \le x \le 2$$

$$= \int_0^2 \int_0^{4-2x} \cdots\, dy\, dx \quad \# \; 0 \le y \le 4-2x$$

$$= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} 1\, dz\, dy\, dx$$

Integrate!

$$\int_0^2 \int_0^{4-2x} \Big[z \Big]_{z=0}^{z=4-2x-y} dy\, dx$$

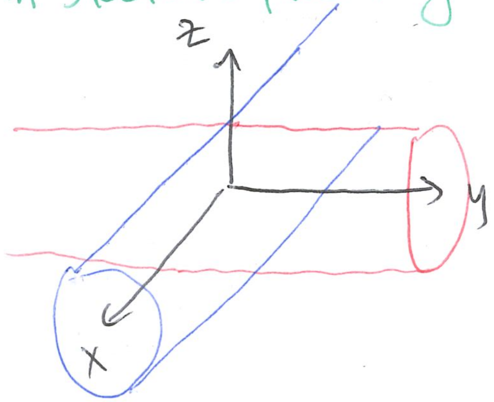
$$= \int_0^2 \int_0^{4-2x} 4 - 2x - y\, dy\, dx = \int_0^2 \left[4y - 2xy - \frac{1}{2}y^2 \right]_{y=0}^{y=4-2x} dx$$

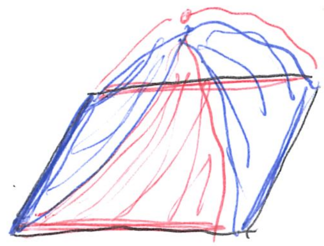
$$= \int_0^2 4(4-2x) - 2x(4-2x) - \frac{1}{2}(4-2x)^2\, dx$$

Example: Two Cylinders Intersecting

Find the volume bounded by the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.

Sketch the region



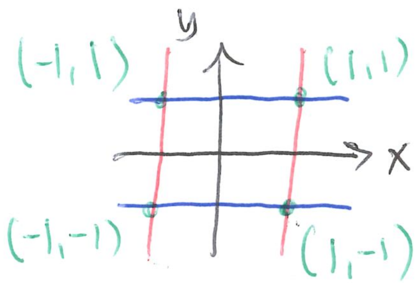


Two blue sides
Two red sides.

Notice this region has nice square slices!

$$V = \int_a^b A(t)\,dt$$

$\downarrow z = 0$



$x^2 + z^2 = 1$
$x^2 + 0 = 1$
$x = \pm 1$

$y^2 + z^2 = 1$
$y^2 + 0^2 = 1$
$y = \pm 1$

Find A(t) of the slice at height z = t.

$z = t \Rightarrow x = \pm\sqrt{1-t^2}$
$y = \pm\sqrt{1-t^2}$

Side length:
$= \sqrt{1-t^2} - (-\sqrt{1-t^2})$
$= 2\sqrt{1-t^2}$

$A(t) = (2\sqrt{1-t^2})(2\sqrt{1-t^2})$
$= 4(1-t^2)$

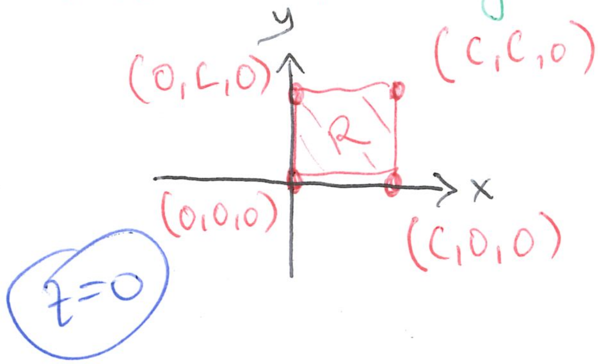
Integrate A(t)

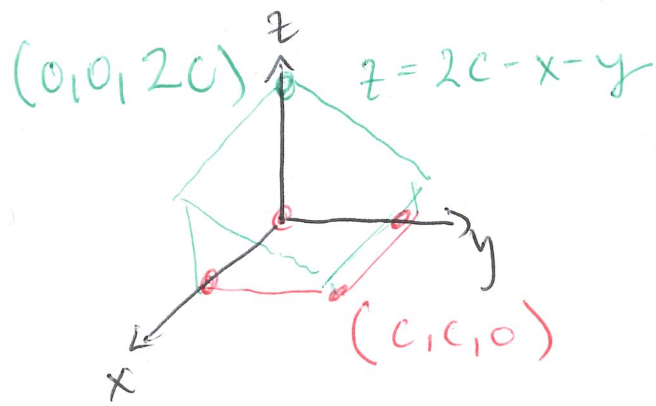
$$V = \int_{-1}^{1} A(t)\,dt = \int_{-1}^{1} 4(1-t^2)\,dt = \left[4(t - \tfrac{1}{3}t^3)\right]_{-1}^{1} = \frac{16}{3}$$

Example: A Variable Volume

Consider the ~~planar~~ region R bounded by $(0,0,0)$, $(C,0,0)$, $(0,C,0)$, $(C,C,0)$.
Find the volume $V(C)$ bounded by $z = 2C - x - y$ above this region.

Sketch the region





Setup the integral

$$V(C) = \iiint_R 1\, dV = \int_0^C \int_0^C \int_0^{2C-x-y} 1\, dz\,dy\,dx$$

$$= \int_0^C \int_0^C \left[z \right]_{z=0}^{z=2C-x-y} dy\,dx = \int_0^C \int_0^C 2C - x - y\, dy\,dx$$

$$= \int_0^C \left[2Cy - xy - \frac{1}{2}y^2 \right]_{y=0}^{y=C} dx$$

$$= \int_0^C 2C^2 - Cx - \frac{1}{2}C^2\, dx = \int_0^C \frac{3}{2}C^2 - Cx\, dx$$

$$= \left[\frac{3}{2}C^2 x - \frac{1}{2}Cx^2 \right]_0^C = \frac{3}{2}C^3 - \frac{1}{2}C^3 = C^3$$

Thus, the volume is $V(C) = C^3$.