Week 11: Coordinates and Transformations

Polar Coordinates (M&T p. 52)

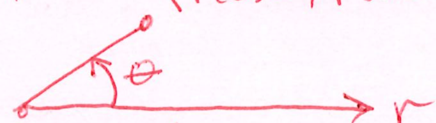
The point with polar coordinates (r, θ) has Cartesian coordinates $(x, y) = (r\cos\theta, r\sin\theta)$.

Example: Some Points in Polar Coordinates

Sketch the following points (r, θ):

1. $\left(1, \frac{\pi}{4}\right)$
2. $\left(1, \frac{\pi}{2}\right)$
3. $\left(\sqrt{2}, -\frac{\pi}{4}\right)$

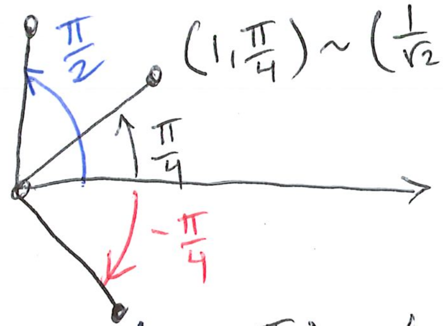
$(r, \theta) \sim (r\cos\theta, r\sin\theta)$



The Polar Axis

$\left(1, \frac{\pi}{2}\right) \sim (0,1)$

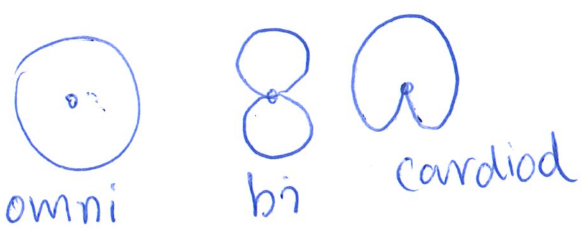
$\left(1, \frac{\pi}{4}\right) \sim \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



$\left(\sqrt{2}, -\frac{\pi}{4}\right) \sim \left(\sqrt{2}\frac{1}{\sqrt{2}}, \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)\right)$

$= (1, -1)$

(!) Pairs of numbers are ambiguous. Be careful to specify polar vs Cartesian.



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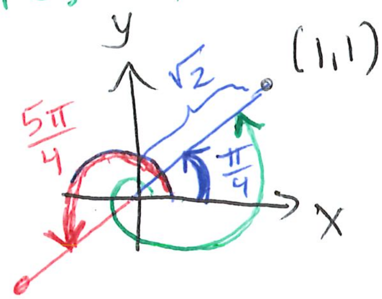
Example: Polar Coordinates are Ambiguous

Find three polar coordinate representations of the point $(x, y) = (1, 1)$.
Warning: There are infinitely many!

Find a single polar representation

$$(r, \theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)$$



$$= \left(-\sqrt{2}, \frac{5\pi}{4}\right)$$

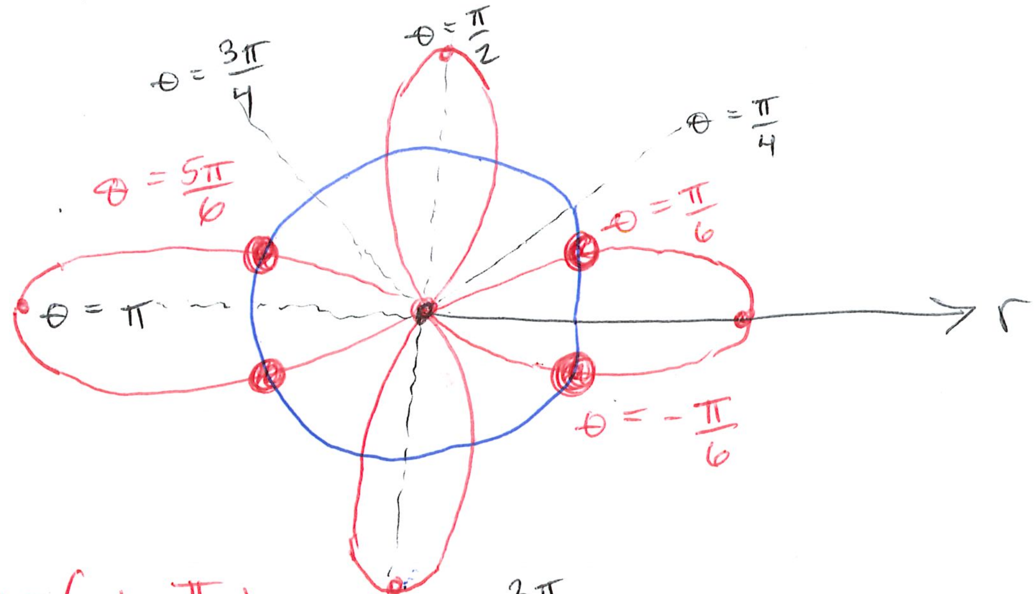
$$= \left(\sqrt{2}, \frac{\pi}{4} + 2\pi\right)$$

- opposite direction and negative r
- 2π-periodicity

Example: Polar Curves

Sketch the polar following polar curves:

- $r = 1/2$ → nice simple circle
- $r = \cos(2\theta)$



$(r, \theta) = (-1, \frac{\pi}{2}) \rightarrow \theta = \frac{3\pi}{2}$

Plot the curve $r = \cos(2\theta)$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	1	0	-1	0	1

Find intersections

$$\cos(2\theta) = \frac{1}{2}$$

$$\Longleftrightarrow 2\theta = \frac{\pi}{3}, -\frac{\pi}{3}$$

$$\Longleftrightarrow \theta = \frac{\pi}{6}, -\frac{\pi}{6}$$

Example: Points of Intersection are Tricky!

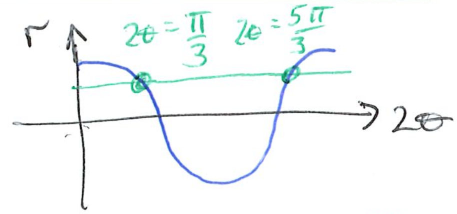
Find all the points of intersection between $r = \cos(2\theta)$ and $r = 1/2$.

Set curves equal

$$\cos(2\theta) = \frac{1}{2}$$

Domain $\cos(t) = [0, 2\pi)$.

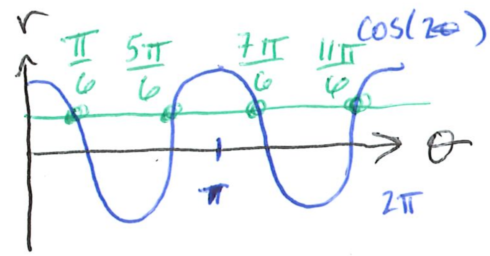
$$\Longleftrightarrow \quad 2\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$



$$2\theta = 2\pi - \frac{\pi}{3} = \frac{6\pi - \pi}{3} = \frac{5\pi}{3}$$

where $0 \le 2\theta \le 2\pi$

$$\Longleftrightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

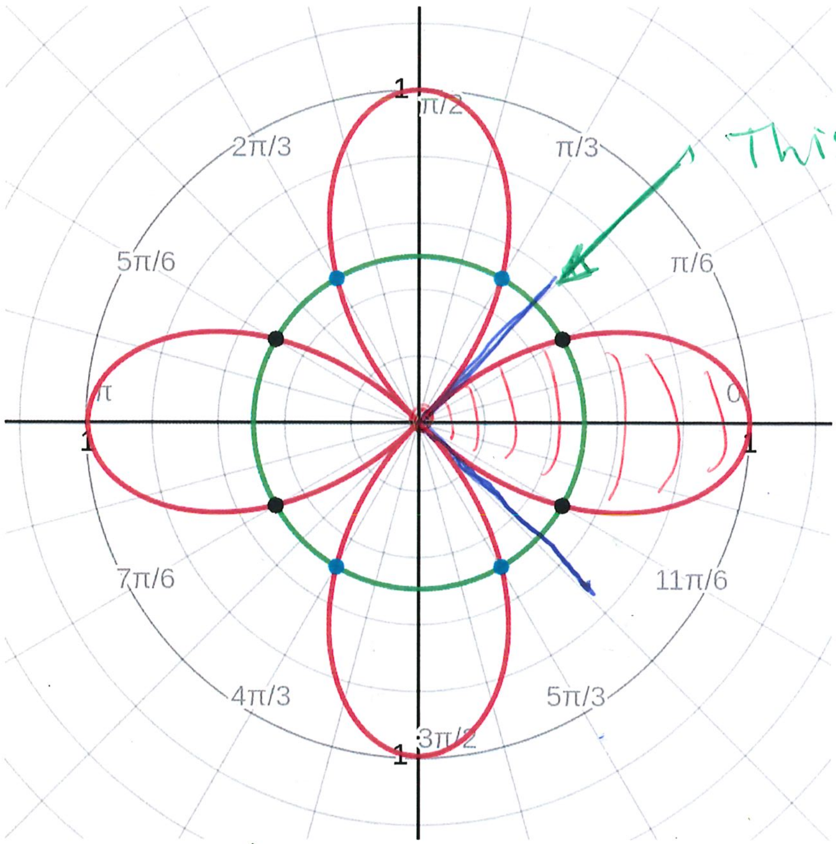


Oh no! We are still missing four points of intersection.

Add π to get the next two

The additional four intersection points come from the circle $r = -\frac{1}{2}$ (which is equivalent to $r = \frac{1}{2}$.)

$$(r,\theta) = \overbrace{\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right)}^{\cos(2\theta) = \frac{1}{2}}, \overbrace{\left(\frac{-1}{2}, \frac{2\pi}{6}\right), \left(\frac{-1}{2}, \frac{4\pi}{6}\right), \left(\frac{-1}{2}, \frac{8\pi}{6}\right), \left(\frac{-1}{2}, \frac{10\pi}{6}\right)}^{\cos(2\theta) = -\frac{1}{2}}$$

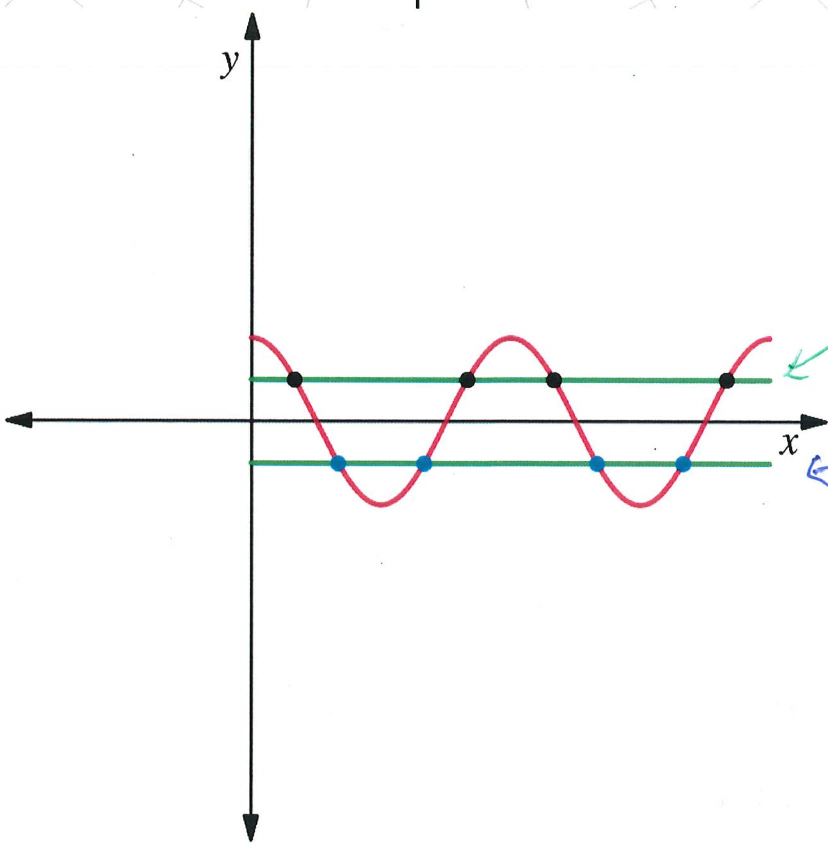


This circle is both $r = \frac{1}{2}$ and $r = -\frac{1}{2}$

Polar Coords

These allow us to build cylindrical and spherical coord.s.

Really, the set us up for the $\mathbb{R}^n$ analogue of u-substitution.



$\cos 2\theta = \frac{1}{2}$

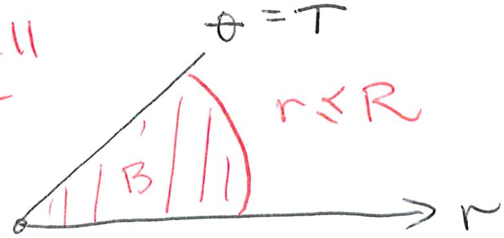
$\cos 2\theta = -\frac{1}{2}$

Activity: Generalize the Integral

- Sketch the region $B = \{(r, \theta) : 0 \le \theta \le T \quad 0 \le r \le R\}$.
- Calculate the area of B by comparing it to a circle of radius R.
- Write a Riemann sum for the area bounded by $r = f(\theta)$ where $a \le \theta \le b$.
- Write an integral formula for the area bounded by $r = f(\theta)$ where $a \le \theta \le b$.

5 min

$\Delta\theta$ is a small change in the angle.



$$\text{Area}(B) = \begin{pmatrix}\text{portion}\\ \text{of circle}\\ \text{inside } B\end{pmatrix}\begin{pmatrix}\text{total area}\\ \text{of circle}\end{pmatrix}$$

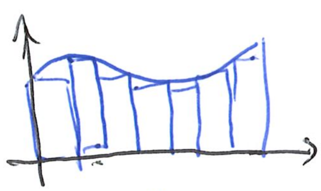
$$= \left(\frac{T}{2\pi}\right)\pi R^2 = \frac{1}{2} T R^2$$

Whoa! The π cancels!

$\frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$

$$\lim_{N \to \infty} \sum_{k=0}^{N-1} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta =$$

$$= \frac{1}{2}\int_a^b [f(\theta)]^2 \, d\theta$$



$y = f(x)$

$$\lim_{N \to \infty} \sum_{k=0}^{N-1} f(x_i^*) \Delta x = \int_a^b f(x)\, dx$$

$f(x_i^*)\Delta x$

Area of Rect.

Example: Find the Area Bounded by a Polar Curve

Calculate the area bounded by one "petal" of the curve $r = \cos(2\theta)$.

Find self-intersections at $r=0$

$$0 = \cos(2\theta)$$
$$\Leftrightarrow 2\theta = \frac{\pi}{2}, -\frac{\pi}{2}$$
$$\Leftrightarrow \theta = \frac{\pi}{4}, -\frac{\pi}{4}$$

Setup the integral

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}\left[\cos(2\theta)\right]^2 d\theta$$
$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta)\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos(2\cdot 2\theta)}{2}\, d\theta$$
$$= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1+\cos(4\theta)\, d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{-\pi/4}^{\pi/4}$$
$$= \frac{1}{4}\left[\left(\frac{\pi}{4} + \frac{1}{4}\sin(\pi)\right) - \left(-\frac{\pi}{4} + \frac{1}{4}\sin(-\pi)\right)\right]$$
$$= \frac{1}{4}\left[\frac{2\pi}{4}\right] = \frac{\pi}{8}$$

Thus, the area of one "petal" is $A = \frac{\pi}{8}$.

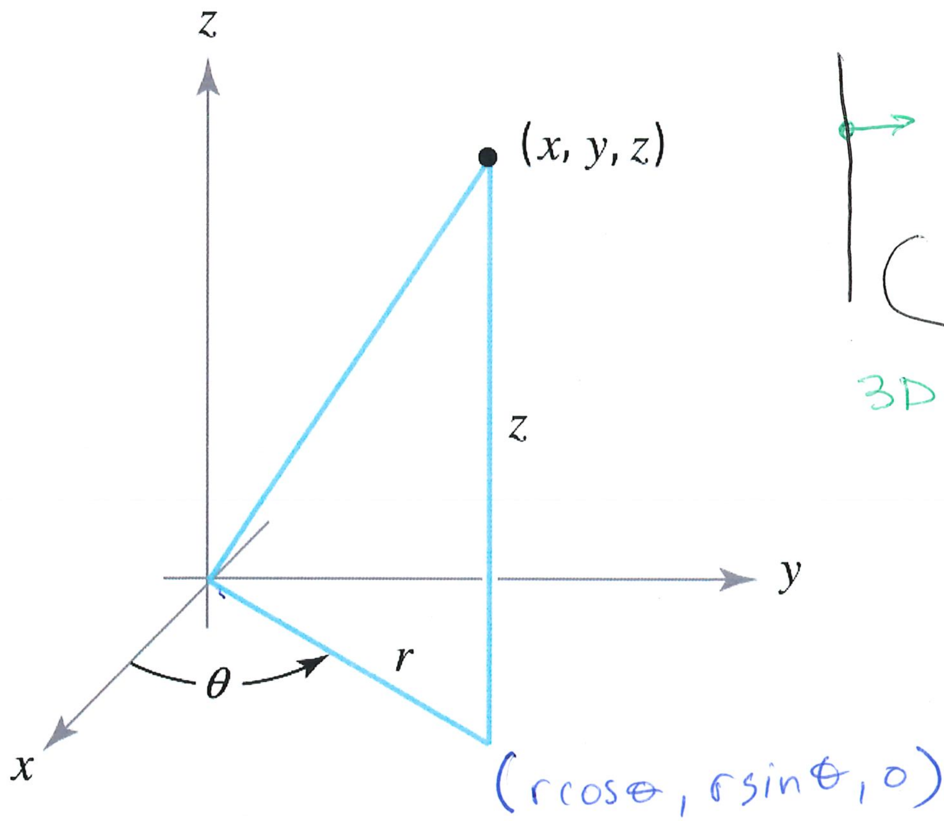
Cylindrical Coordinates (M&T p. 52)

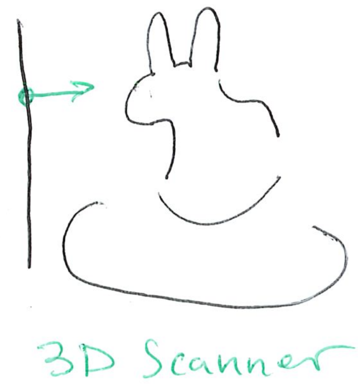
A point with cylindrical coordinates (r, θ, z) has Cartesian coordinates

$$(x, y, z) = (r\cos\theta, r\sin\theta, z).$$

Idea: Cylindrical coordinates are "polar × cartesian." $\mathbb{R}^2 \times \mathbb{R}$

Figure 1.4.2 from M&T p. 53



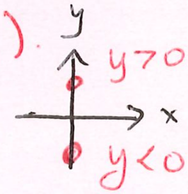


Example: Point to Cylindrical

Convert the ~~point~~ following (cartesian) points to cylindrical coordinates:

1. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
2. $(1, 0, 2)$
3. $(1, -1, 2)$

What happens when $x=0$ in cartesian?
Look at the point $(0, y)$.
$y>0 \Longrightarrow \theta = \pi/2$
$y<0 \Longrightarrow \theta = 3\pi/2$



Convert 1.

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \Longrightarrow z = \frac{1}{\sqrt{2}}$$

$$r = \sqrt{x^2+y^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1$$

$$\theta = \arctan\left(y/x\right) = \arctan(1) = \frac{\pi}{4}$$

Thus, $(r, \theta, z) = \left(1, \frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

Convert 2.

$$(1, 0, 2) \Longrightarrow z = 2$$

$$r = \sqrt{x^2+y^2} = \sqrt{1+0} = \sqrt{1} = 1$$

$$\theta = \arctan\left(y/x\right) = \arctan(0) = 0$$

Thus, $(r, \theta, z) = (1, 0, 2)$

Whoa! It is the same in cylindrical and cartesian!

Exercise: Convert 3.

Example: Common Surfaces in Cylindrical

Convert the following surfaces to cylindrical coordinates:

Cone $z^2 = x^2 + y^2$

Paraboloid $z = x^2 + y^2$

Sphere $x^2 + y^2 + z^2 = 1$

Convert the cone.

$$z^2 = x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$
$$= r^2[\cos^2\theta + \sin^2\theta] = r^2$$
$$\Leftrightarrow z^2 = r^2$$

Convert paraboloid.

$$z = x^2 + y^2 = r^2$$

$r^2 = x^2 + y^2$

Convert sphere

$$x^2 + y^2 + z^2 = 1$$
$$\Leftrightarrow r^2 + z^2 = 1.$$

All of these examples do not include θ. Thus, the resulting graphs will be rotationally symmetric.

Eventually, our double integrals over these regions will be easy to evaluate.

 $(\rho, \theta, \phi) =$ "(rho, theta, phi)"

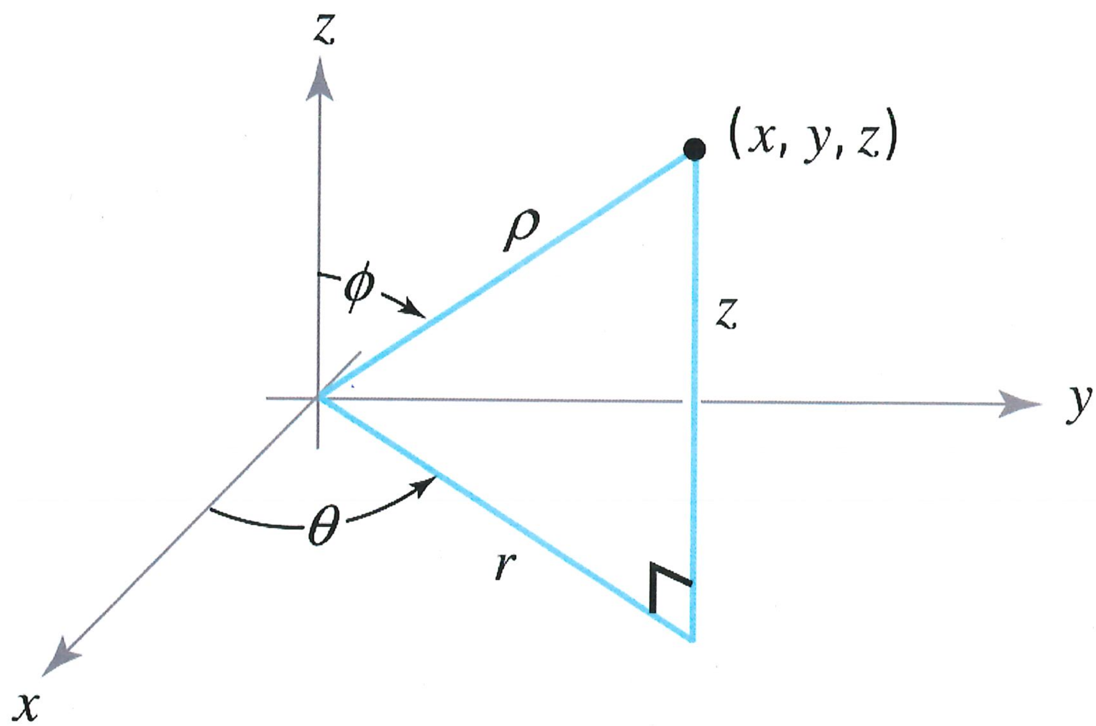
Spherical Coordinates (M&T p. 54)

The spherical coordinates of a point with Cartesian coordinates (x, y, z) are (ρ, θ, ϕ) such that:

$$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$

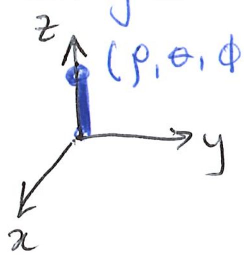
where $0 \leq \rho$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$. *Mnemonic:* ϕ is for z-axis and has a vertical line in it.

Figure 1.4.5 from M&T p. 55



ϕ measures angle of point to z-axis.

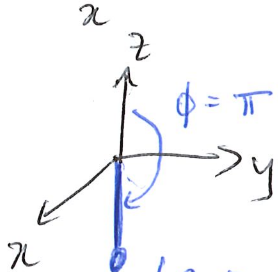
$\phi = 0 \implies$



$(\rho, \theta, \phi) = (\rho, \theta, 0)$

"straight up"

$\phi = \pi \implies$



"straight down"

$(\rho, \theta, \phi) = (\rho, \theta, \pi)$

Example: Cartesian to Spherical

Convert the following points (x, y, z) to spherical coordinates.

1. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
2. $(1, 0, 0)$
3. $(1, 1, 0)$

Convert 1

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = \sqrt{\frac{3}{2}}$$

Notice that $x=y=z$ for $(x,y,z) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

We guess: $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$. ← oops!

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan(1) = \frac{\pi}{4}.$$

$$z = \rho \cos\phi$$

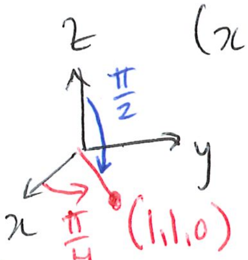
$$\Leftrightarrow \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}\cos\phi$$

$$\Leftrightarrow \frac{1}{\sqrt{3}} = \cos\phi$$

$$\Leftrightarrow \phi = \arccos\left(\frac{1}{\sqrt{3}}\right).$$

Convert 3

$(x,y,z) = (1,1,0)$

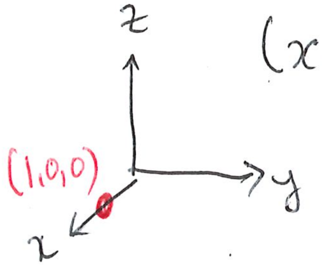


$\rho = \sqrt{2} = \sqrt{x^2+y^2+z^2}$

$\boxed{\phi = \frac{\pi}{2}}$

$\boxed{\theta = \frac{\pi}{4}}$

Convert 2



$(x,y,z) = (1,0,0) \Rightarrow \rho = 1 \quad \phi = \frac{\pi}{2} \quad \theta = 0$

See reverse. →

Transformations (M&T p. 308)

Let $D^* \subset \mathbb{R}^2$. A C^2-function $T : D^* \to \mathbb{R}^2$ is a transformation (or map) of the plane.
We write: $(x, y) = T(x^*, y^*)$ to distinguish the output (x, y) from the input (x^*, y^*).

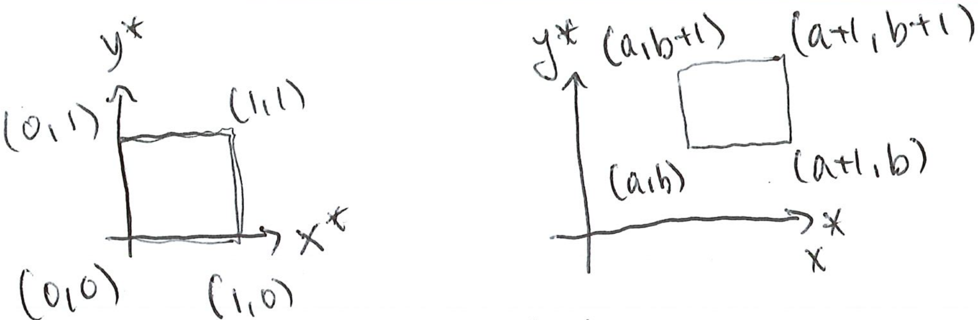
Example: Common Transformations

Translation For any $(a, b) \in \mathbb{R}^2$ we have a transformation given by $T(x^*, y^*) = (x^* + a, y^* + b)$.

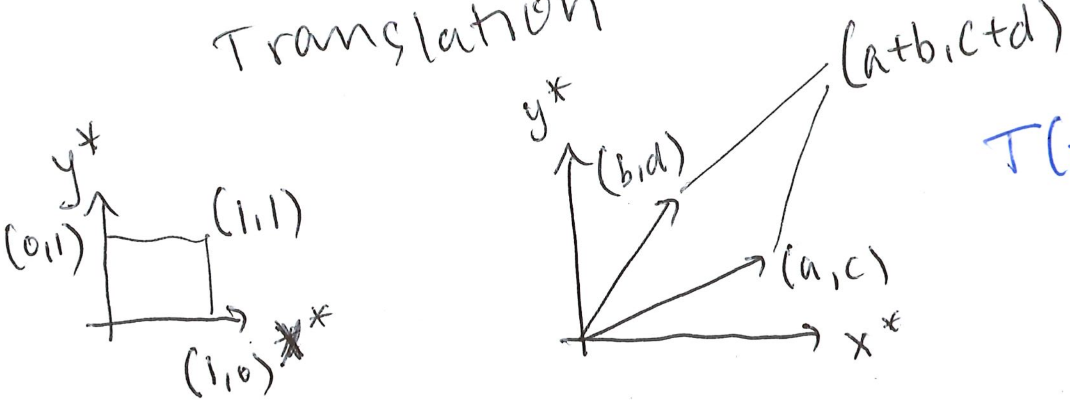
Linear Maps For a 2×2 matrix M, we get a transformation $T(x^*, y^*) = M \begin{bmatrix} x^* \\ y^* \end{bmatrix}$.

Polar to Cartesian We can transform polar coordinates (r, θ) in to cartesian coordinates:

$$T(r, \theta) = (r\cos\theta, r\sin\theta).$$



Translation



$T(x^*, y^*) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix}$

Linear Maps

! Here we use $[0,1] \times [0, \frac{\pi}{2}]$ to avoid terms like $\cos(1)$ and $\sin(1)$.

Polar Coords

One-to-One and Onto **(M&T p. 310)**

A map $T : D^* \longrightarrow D$ is <u>one-to-one</u> (or <u>injective</u>) if $T(\mathbf{x}) = T(\mathbf{x}')$ implies $\mathbf{x} = \mathbf{x}'$.
A map $T : D^* \longrightarrow D$ is <u>onto</u> (or <u>surjective</u>) if for all $\mathbf{x} \in D$ there is $\mathbf{x}^* \in D^*$ such that: $T(\mathbf{x}^*) = \mathbf{x}$.

If a transformation is one-to-one and onto then we say that it is <u>bijective</u>.

Activity: Bijectivity for Linear Maps

When is a linear transformation $T(x^*, y^*) = M \begin{bmatrix} x^* \\ y^* \end{bmatrix}$ bijective?

If $T: \mathbb{R}^n \to \mathbb{R}^n$ then
onto $\iff$ one-to-one

If M invertible then T is bijective.

If $\det(M) \neq 0$ then T is bijective.

We will see det later
when we do change of coordinates

Example: Translation is One-to-One and Onto

Consider a translation transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T(x^*, y^*) = (a + x^*, b + y^*)$. Show that T is one-to-one and onto.

Show that T is one-to-one.

Suppose $T(x_1^*, y_1^*) = T(x_2^*, y_2^*)$.

This gives: $(x_1^* + a, y_1^* + b) = (x_2^* + a, y_2^* + b)$

Thus $\begin{cases} x_1^* + a = x_2^* + a \\ y_1^* + b = y_2^* + b \end{cases} \Longleftrightarrow \begin{cases} x_1^* = x_2^* \\ y_1^* = y_2^* \end{cases}$

It follows $(x_1^*, y_1^*) = (x_2^*, y_2^*)$.

Show that T is onto.

Pick $(x_2^*, y_2^*) \in \mathbb{R}^2$. ← Pick an arbitrary point in co-domain.

Consider $(x_1^*, y_1^*) = (x_2^* - a, y_2^* - b) \in \mathbb{R}^2$

We have:

$$T(x_1^*, y_1^*) = \big((x_2^* - a) + a, (y_2^* - b) + b \big)$$

$$= (x_2^*, y_2^*)$$ ← Map something to the point picked previously

Therefore, T is one-to-one and onto.

Example: Polar Coordinates are Onto but not One-to-One

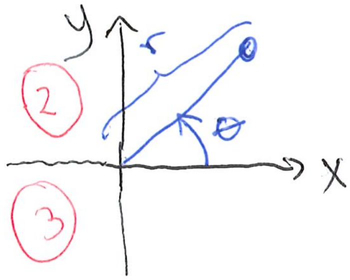
Consider a translation transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Show that T is onto but not one-to-one.

Show T is not one-to-one

$(0,0) = T(0,\theta)$ for all θ.

For example, $(0,0) = T(0,0) = T(0,2\pi)$ but $(0,0) \neq (0,2\pi)$

Show that T is onto.



" Clearly, $(x,y) = (r\cos\theta, r\sin\theta)$."

If $x \neq 0$ then pick $\theta = \arctan\left(\frac{y}{x}\right)$ and $r = \sqrt{x^2+y^2}$

If $x = 0$ then either $y \geq 0$ or $y < 0$.

When $y \geq 0$ we pick $\theta = \frac{\pi}{2}$ and $r = y$.

When $y < 0$ we pick $\theta = -\frac{\pi}{2}$ and $r = -y$.

Extend this to cover (x,y) in Quadrants 2 and 3.

Follow-Up Question: What restricted domain $D^* \subset \mathbb{R}^2$ would make T one-to-one and onto $\mathbb{R}^2$?

Pick $-\pi < \theta \leq \pi$ and $0 < r$ $\quad D^* = (0,\infty) \times (-\pi, \pi]$

We set:

$D^* = \{(0,0)\} \cup (0,\infty) \times (-\pi, \pi]$