Week 12: Change of Variables

Single Variable Substitution (M&T p. 318)

If f is continuous and $x(u)$ is C^1 on $[a,b]$ then:

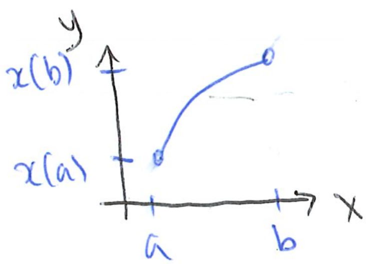
$$\int_a^b f(x(u))\frac{dx}{du}\,du = \int_{x(a)}^{x(b)} f(x)\,dx$$

If $x(u)$ is one-to-one on $[a,b]$ then either: x is strictly increasing or strictly decreasing.

$$\frac{dx}{du} > 0 \quad \text{or} \quad \frac{dx}{du} < 0$$ by Rolle's Theorem

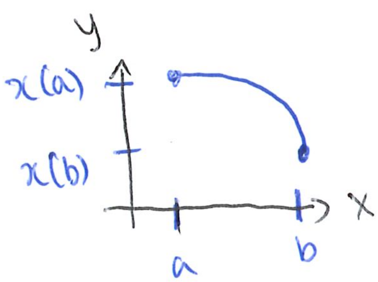
We let $I^* = [a,b]$ and $I = [x(a), x(b)]$ (in the case when x is increasing) or $I = [x(b), x(a)]$ (in the case when x is decreasing). With this notation substitution becomes:

$$\boxed{\int_{I^*} f(x(u))\left|\frac{dx}{du}\right|\,du = \int_I f(x)\,dx}$$



In the increasing picture: $a \leq b$ and $x(a) \leq x(b)$.

We get: $\frac{dx}{du} = \left|\frac{dx}{du}\right|$ and so $\int_{I^*} f(x(u))\left|\frac{dx}{du}\right| du = \int_I f(x)dx$



In the decreasing picture: $a \leq b$ and $x(b) \leq x(a)$

$$\int_{I^*} f(x(u))\frac{dx}{du}\,du = \int_{x(b)}^{x(a)} f(x(u))\frac{dx}{du}\,du$$

FTC gives $(-)$ $\rightarrow$

$$= -\int_{x(a)}^{x(b)} f(x(u))\frac{dx}{du}\,du$$

$$= \int_{I^*} f(x(u))\left|\frac{dx}{du}\right|\,du$$

The uv-Plane (M&T p. 314)

A transformation $T : D^* \to D$ can be written:

These are 2-variable functions

$$T(u,v) = (x(u,v), y(u,v)) = (x, y)$$

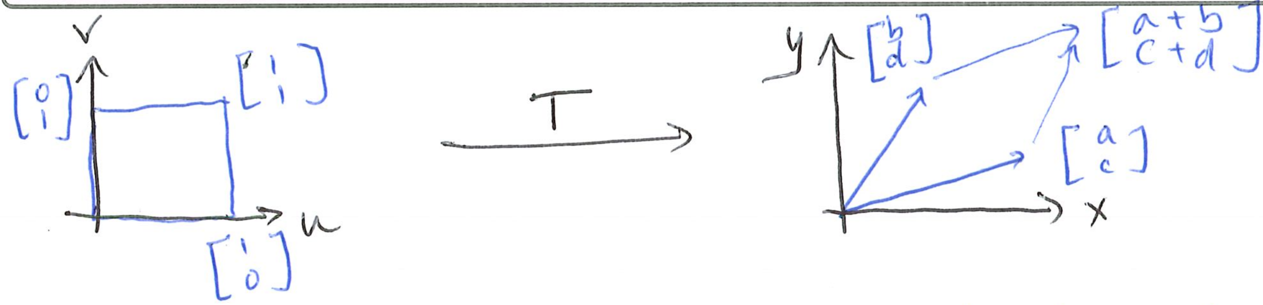
where $D^* \subseteq \{(u,v) : u, v \in \mathbb{R}\} = \mathbb{R}^2$ and $D \subseteq \{(x,y) : x, y \in \mathbb{R}\} = \mathbb{R}^2$.

Example: An Argument for Change of Coordinates

How does a linear map deform area? Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Compute:

$$\text{Area}([0,1]^2) \qquad \text{Area}(T([0,1]^2))$$

Moreover, what is the derivative $\mathbf{D}T$?



$\text{Area}([0,1]^2)$
$= 1 \cdot 1 = 1$

$\text{Area}(T([0,1]^2))$
$= \left| \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|$
$= |ad - bc|$

! Area ought to be positive.

What is DT?

$$DT = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

! A linear map has its matrix as its derivative. And we know how matrices deform area.

The Jacobian Determinant (M&T p. 315)

For a C^1-transformation $T(u,v) = (x(u,v), y(u,v))$ we define the Jacobian determinant to be:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det(\mathbf{D}T) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}.$$

Note: We usually use the absolute value $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

Theorem: Change of Variables

If D and D^* are elementary regions in the plane, and $T : D^* \to D$ is C^1 and one-to-one on D^* then for any integrable $f : D \to \mathbb{R}$ we have:

$$\iint_D f(x,y)\, dxdy = \iint_{D^*} f(x(u,v), y(u,v)) \boxed{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv}$$

This term tracks how area changes under T.

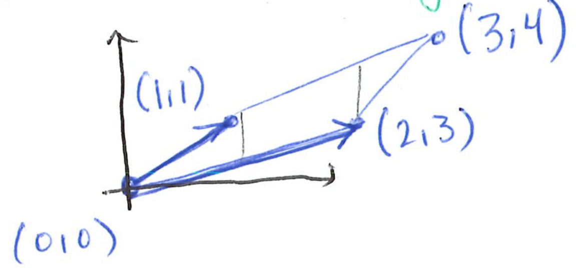
Notice: This looks a lot like the fancy re-writing on substitution:

$$\int_I f(x)\, dx = \int_{I^*} f(x(u)) \left|\frac{dx}{du}\right| du$$

Example: Simplify a Region

Consider the region with vertices $(0,0)$, $(1,1)$, $(2,3)$, $(3,4)$. Sketch the region, and find a linear transformation T such that: $T([0,1]^2) = R$. Evaluate $\iint\limits_R xy\, dA$ using change of coordinates.

Sketch the region



Integrating over this region would suck! We would have three changes in our y-bounds.

Find a linear transformation

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Alternatively $T = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ also works.

Apply change of coords

$$\iint\limits_R xy\,dA = \iint\limits_{R^* = [0,1]^2} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

We have: $x = 2u+v \quad y = 3u+v$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = |DT| = \left| \det \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \right| = |-1| = 1$$

$$\iint\limits_R xy\,dA = \int_0^1 \int_0^1 (2u+v)(3u+v)\, 1\, du\,dv$$

$$= \int_0^1 \int_0^1 6u^2 + 5uv + v^2\, du\,dv \leftarrow \text{polynomial!}$$

Example: Integration in Polar Coordinates

Find the Jacobian of the polar coordinates transformation $T(r, \theta) = (r\cos\theta, r\sin\theta)$.

\# Compute $\dfrac{\partial(x,y)}{\partial(r,\theta)}$

$$\det DT = \det \frac{\partial(x,y)}{\partial(r,\theta)}$$ ← order matters

$$= \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\boxed{dA = |r|\,dr\,d\theta}$$

What do polar pirates say? Arrr-$rdrd\theta$!

Example: An Integral That's Nicer in Polar Coordinates

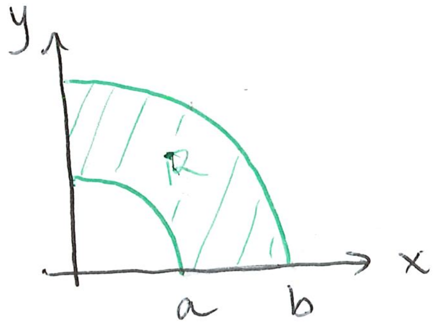
Evaluate

Let $a, b > 0$.

$$\iint_R e^{x^2+y^2}\, dA$$

where $R = \{(x,y) : 0 \le x, y \text{ and } a^2 \le x^2+y^2 \le b^2\}$. *Note:* $dA = r\, dr d\theta$.

Sketch the region



Convert to polar

$$R^* = \left\{ (r,\theta) : \begin{array}{l} 0 \le \theta \le \frac{\pi}{2} \\ a \le r \le b \end{array} \right\}$$

$$= [a,b] \times [0, \tfrac{\pi}{2}]$$

Apply change of variables

$$\iint_R e^{x^2+y^2} \boxed{dA} = \int_0^{\frac{\pi}{2}} \int_a^b e^{r^2} \boxed{r\, dr\, d\theta}$$

How do we get $x^2+y^2 = r^2$?

$(x,y) = (r\cos\theta, r\sin\theta)$

$x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2 \cdot 1^2 = r^2$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_{a^2}^{b^2} e^u\, du\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[e^{b^2} - e^{a^2}\right] d\theta$$

$$= \frac{\pi}{4}\left[e^{b^2} - e^{a^2}\right]$$

Change of Variables of Cylindrical Coordinates (M&T p. 325)

Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$ of the cyclindrical coordinates transformation: "$\underbrace{\text{polar}}_{r} \times \underbrace{\text{cartesian}}_{1}$"

$$T(r,\theta,z) = (r\cos\theta, r\sin\theta, z).$$

Compute Jacobian

$$\det \frac{\partial(x,y,z)}{\partial(r,\theta,z)} \leftarrow$$ The order of these coordinates matters.

$$= \det \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix}$$

$$= \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow$$ Expend along this row

$$= 0 - 0 + 1 \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= 1 \cdot r = r$$

$$dV = |r|\, dr\, d\theta\, dz$$

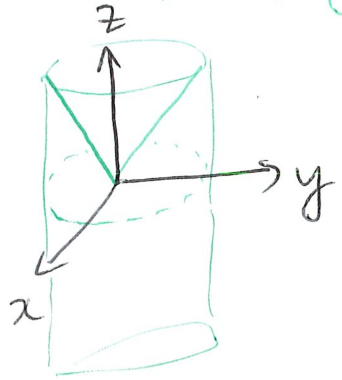
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \frac{\partial(x,y,z)}{\partial(z,r,\theta)}$$ by det rules.

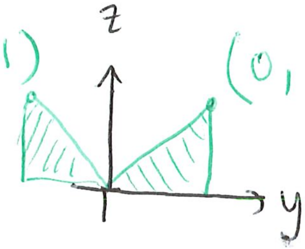
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

Example: A Cylindrical Integral

Use cylindrical coordinates to calculate $\iiint_R z \, dxdydz$ where R is the region inside $x^2 + y^2 = 1$ satisfying $0 \leq z \leq \sqrt{x^2+y^2}$. *Note*: $dV = \cancel{r\, dzdrd\theta}$. $r\, drd\theta dz$

Sketch the region.





Describe R in cylindrical coords.

$$R^* = \left\{ (r,\theta,z) : \begin{matrix} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq r \end{matrix} \right\}$$

Apply change of coords

$$\iiint_R z \, dxdydz = \iiint_{R^*} z \; r\, dzdrd\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^r zr \, dzdrd\theta$$

Double check order of Jacobian.

$r\, dzdrd\theta$ vs $r\, drd\theta dz$ against book.

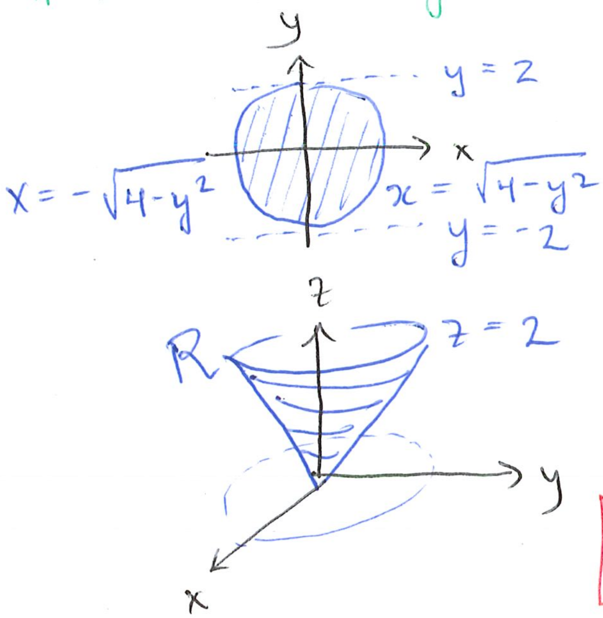
Example: A Nasty Triple Integral

Evaluate by changing to cyclindrical coordinates.

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\ dzdxdy$$

Let $f(x,y,z) = xz$.
Notice: "odd"
$f(x,y,z) = -f(-x,y,z)$
and R is symmetric.

Sketch the region



We note:

$$\sqrt{x^2+y^2} \leq z \leq 2$$

Setup region in cylindrical.

$$R^* = \left\{ (r,\theta,z) : \begin{array}{l} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ r \leq z \leq 2 \end{array} \right\}$$

$$\boxed{(x,y,z) = (r\cos\theta, r\sin\theta, z)}$$

$$\iiint_R xz\ dzdxdy = \iiint_{R^*} (r\cos\theta) z\, r\, dzdrd\theta$$

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2\cos\theta\, z\ dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}z^2 r^2\cos\theta\right]_{z=r}^{z=2} dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{1}{2}(2^2-r^2)r^2\cos\theta\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 2r^2\cos\theta - \frac{1}{2}r^4\cos\theta\ dr\,d\theta$$

Change of Variables for Spherical Coordinates (M&T p. 324)

Compute the Jacobian $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$ of the spherical coordinates transformation:

$$T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

\# compute det $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$

$$\det DT = \det \begin{bmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{bmatrix}$$

$$= \det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\,\cancel{\cos}\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

$$= \cos\phi \det\begin{bmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{bmatrix}$$

$$- \rho\sin\phi \det\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\cos\theta & \rho\sin\phi\cos\theta \end{bmatrix}$$

$$= \rho^2\cos\phi \begin{bmatrix} -\sin\phi\sin\theta & \cos\phi\cos\theta \\ \sin\phi\cos\theta & \cos\phi\sin\theta \end{bmatrix}$$

$$- \rho\sin^3\phi \det\begin{bmatrix} \cos\theta & \rho\sin\theta \\ \cos\theta & \rho\cos\theta \end{bmatrix}$$

To be continued.

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\rho^2\sin\theta$$

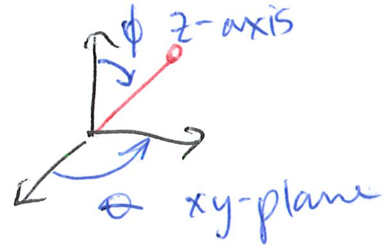
Example: The Volume of a Sphere

Compute the volume of a sphere of radius r. *Note*: $dV = \rho^2 \sin\phi \; d\rho d\theta d\phi$.

\# Setup in spherical coords.

$$S = \{(\rho, \theta, \phi) : 0 \le \rho \le r \quad 0 \le \theta \le 2\pi \quad 0 \le \phi \le \pi\}$$

$$= [0, r] \times [0, 2\pi] \times [0, \pi]$$



\# Setup volume integral

$$V = \iiint_S dV = \int_0^\pi \int_0^{2\pi} \int_0^r \rho^2 \sin\phi \; d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \left[\frac{1}{3}\rho^3 \sin\phi\right]_{\rho=0}^{\rho=r} d\theta \, d\phi = \int_0^\pi \int_0^{2\pi} \frac{1}{3} r^3 \sin\phi \; d\theta \, d\phi$$

$$= \int_0^\pi \left[\frac{1}{3} r^3 \theta \sin\phi\right]_{\theta=0}^{\theta=2\pi} d\phi$$

$$= \int_0^\pi \frac{2\pi}{3} r^3 \sin\phi \; d\phi = \left[-\frac{2\pi}{3} r^3 \cos\phi\right]_{\phi=0}^{\phi=\pi}$$

$$= \frac{4\pi}{3} r^3$$

$$V = \frac{4}{3}\pi r^3$$

Example: An Interesting Average

What is the average distance of a point in a sphere of radius r to its center?

setup the integral

$$A = \frac{1}{\mathrm{vol}(S)} \iiint_S \sqrt{x^2+y^2+z^2}\, dV$$

Switch to spherical coords $\rho^2 = x^2+y^2+z^2$

$$= \frac{1}{\mathrm{vol}(S)} \int_0^{\pi}\int_0^{2\pi}\int_0^{r} \sqrt{\rho^2}\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

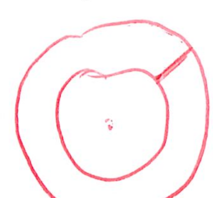
$$= \frac{1}{\mathrm{vol}(S)} \int_0^{\pi}\int_0^{2\pi}\int_0^{r} \rho^3 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \frac{1}{\mathrm{vol}(S)} \int_0^{\pi}\int_0^{2\pi} \left[\frac{1}{4}\rho^4 \sin\phi\right]_{\rho=0}^{\rho=r} d\theta\, d\phi$$

$$= \frac{1}{\mathrm{vol}(S)} \int_0^{\pi}\int_0^{2\pi} \frac{1}{4} r^4 \sin\phi \, d\theta\, d\phi$$

$$= \frac{1}{\mathrm{vol}(S)} \int_0^{\pi} \frac{2\pi}{4} r^4 \sin\phi \, d\phi = \frac{1}{\mathrm{vol}(S)} \pi r^4$$

$$= \frac{1}{\frac{4\pi}{3} r^3} \pi r^4 = \frac{3}{4} r$$



Pizzas and areas?

Why not $\frac{1}{2} r$?

Example: A Spherical Integral

Evaluate the integral $\iiint\limits_R (x^2+y^2+z^2)^2\, dV$ and $R = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$.

$= \underset{\rho}{[0,1]} \times \underset{\theta}{[0,2\pi]} \times \underset{\phi}{[0,\pi]}$

$$\iiint\limits_R (x^2+y^2+z^2)^2\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \underbrace{(\rho^2)^2}_{(x^2+y^2+z^2)^2}\, \underbrace{\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}_{dV}$$

Exercise: Calculate this integral.

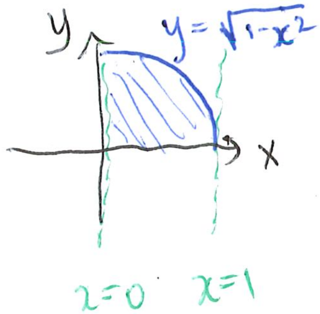
Example: A Nasty Triple Integral

Evaluate by switching to spherical coordinates.

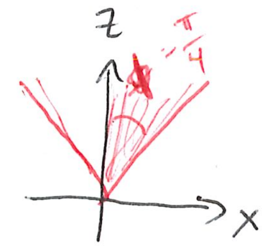
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \; dz dy dx$$

Sketch region of integration

Look at $z=0$ plane.



In polar coords, this is $0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le 1$.



The cone is $0 \le \phi \le \frac{\pi}{4}$.

In 3D our z-values are bounded by

$z = \sqrt{x^2+y^2}$ ⟵ cone angle $\phi = \frac{\pi}{4}$ with z-axis

$z = \sqrt{2-x^2-y^2}$ ⟵ sphere of radius $\sqrt{2}$.

Swith to spherical coords

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} (x)(y)\rho^2 \sin\phi \; d\rho \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)\rho^2 \sin\phi \; d\rho \, d\theta \, d\phi$$

Example: The Gaussian Integral

Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. *Note*: This is a very clever computation, a true gem of mathematics.

Observe: you cannot integrate $\int e^{-x^2} dx$.

Whoa!

$$\iint e^{-(x^2+y^2)}\,dA = \left[\iint_A e^{-x^2}\,dA\right]\left[\iint_A e^{-y^2}\,dA\right]$$

Integrating this over the whole plane we get:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \lim_{t\to\infty}\int_0^{2\pi}\int_0^{t} e^{-r^2}\,r\,dr\,d\theta$$

$u = r^2$
$du = 2r\,dr$

$$= \lim_{t\to\infty}\frac{1}{2}\int_0^{2\pi}\int_0^{t^2} e^{-u}\,du\,d\theta$$

$$= \lim_{t\to\infty}\frac{1}{2}\int_0^{2\pi}\left[-e^{-u}\right]_{u=0}^{u=t^2} d\theta$$

$$= \lim_{t\to\infty}\frac{1}{2}\cdot 2\pi\left(-e^{-t^2}+e^{-0}\right) = \pi(0+1) = \pi$$

We get: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

Summary of MAT B41

- Linear algebra and functions of several variables
- Differentiation of functions $f : \mathbb{R}^n \to \mathbb{R}^k$
- Multivariate chain rule
- Quadratic forms, Hessian matrices, and extrema of multivariate functions
- Constrained optimization and Lagrange multipliers
- Approximation by Taylor polynomials
- Integration of function $f : \mathbb{R}^n \to \mathbb{R}$
- Change of Coordinates: spherical, cylindrical and polar coordinates

Summary of MAT B42

- Week 1: Fourier Series from Hughes-Hallett
- Week 2: Applications Fourier Series to PDEs
  - Heat Equation
  - Wave Equation
- Week 3: Parametric Equations
  - §2.4 Introduction to Paths and Curves
  - §4.1 Acceleration and Newton's Second Law
  - §4.2 Arc Length
- Week 4: Vector Fields
  - §4.3 Vector Fields
  - §7.1 The Path Integral
- Week 5: Line Integrals
  - §7.2: Line Integrals
  - §8.1: Green's Theorem (without the vector form)
- Week 6: Vector Valued Functions
  - §4.4 Divergence and Curl
  - §8.1 Green's Theorem (without the vector form)
- Week 7: Surfaces (Part 1)
  - §7.3 Parametrized Surfaces
  - §7.4 Area of a Surface
- Week 8: Surfaces (Part 2)
  - §7.4 Area of a Surface
  - §7.5: Integrals of Scalar Functions over Surfaces
- Week 9: The Classical Theorems of Vector Calculus (Part 1)
  - §8.2: Stokes' Theorem
  - §8.3: Conservative Vector Fields
- Week 10: The Classical Theorems of Vector Calculus (Part 2)
  - §8.4: Gauss' Theorem
- Week 11: Introduction to Differential Forms
  - §8.5: Differential Forms

