Uniform Convergence

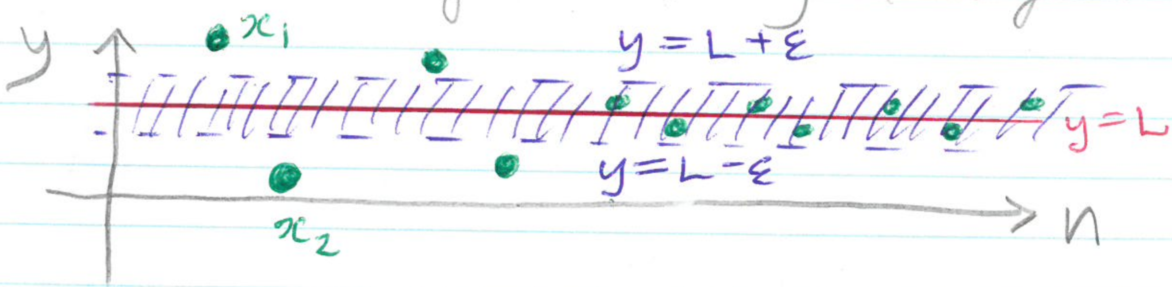
Recall, from MAT A31, the definition of a convergent sequence:

$x_n \longrightarrow L$ if: for all $\varepsilon > 0$ there is N such that

$$n \geq N \Rightarrow |x_n - L| < \varepsilon.$$

Question: What does this mean? Physically? graphically?

"The points x_n get close to L as n gets very large."



Question: Which green dot is x_N?

Ans: x_5 because it is in the ε strip.

We understand how real numbers x_n can limit to L.

Question: What does it mean to say

$$f_n \longrightarrow F$$

for functions f_n?

We start:

$\boxed{f_n \to F}$ if: for all $\varepsilon > 0$
there is N such that

$$n \geq N \Rightarrow \boxed{\qquad\qquad < \varepsilon}$$

Question: How should we fill in this box?

We want something which says "f is close to F".

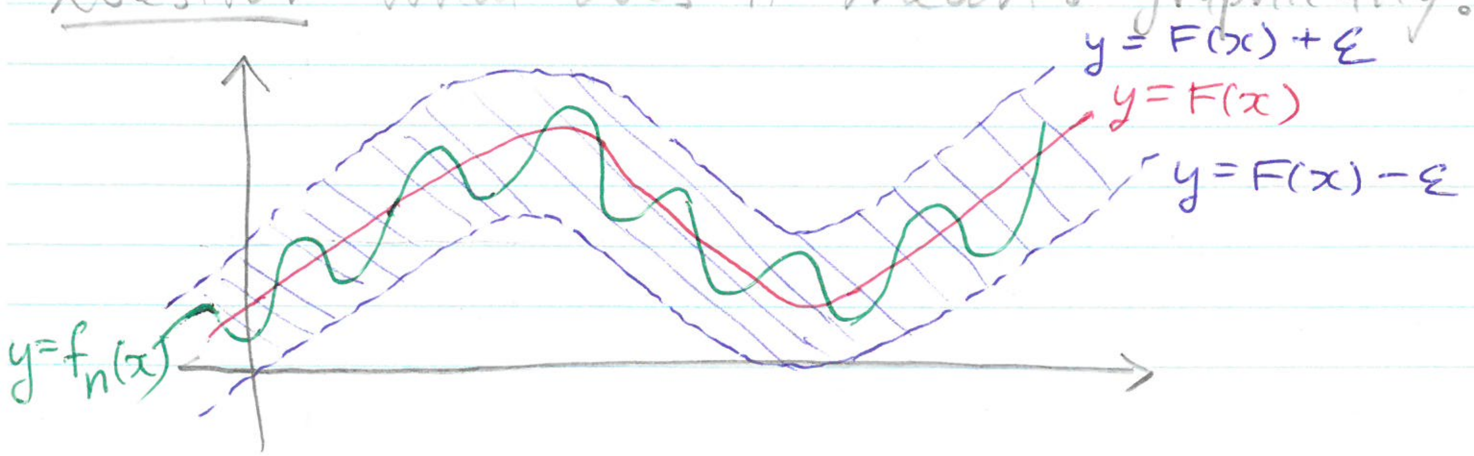
Defn: We say: $f_n \to F$ **uniformly** if:

For all $\varepsilon > 0$ there is N such that:

$$n \geq N \Rightarrow \boxed{\begin{array}{c} |f_n(x) - F(x)| < \varepsilon \\ \text{for all values of } x \end{array}}$$

? There are other interesting definitions that we could make for "$f_n \to F$". We will work with this uniform notion due to Weierstrass.

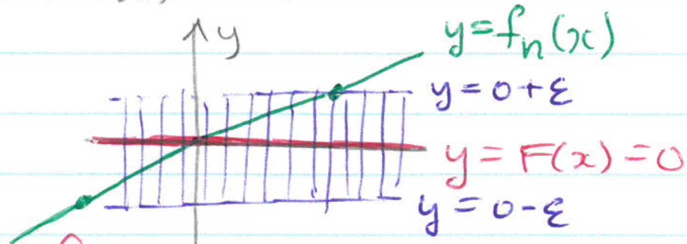
Question: What does it mean? Graphically?



Ex: $f_n(x) = \frac{1}{n}$ (for all x) converges uniformly to $F(x) = 0$.

This agrees with our idea $\frac{1}{n} \to 0$.

Non-Ex: $f_n(x) = \frac{1}{n} x$ does NOT converge uniformly to $F(x) = 0$.



(!) $y = f_n(x)$ leaves the ε-strip.

Theorem:
If f_n is continuous for all n and $f_n \to F$ uniformly then F is continuous.

We need to show:
For all $\varepsilon > 0$ there is $\delta > 0$ such that: $|x - y| < \delta \Rightarrow |F(x) - F(y)| < \varepsilon$.

We have:

- Uniform Convergence
  For all $\varepsilon_F > 0$ there is N_F such that:
  $n \geq N_F \Rightarrow |f_n(x) - F(x)| < \varepsilon_N$
- Continuity
  For all $\varepsilon_k > 0$ there is $\delta_k > 0$
  $|x - y| < \delta_k \Rightarrow |f_k(x) - f_k(y)| < \varepsilon_k$

Question: How do we put these together?

Observations:

Introduce f_k terms.

$$|F(x)-F(y)| = \left| \begin{array}{c} F(x)-f_k(x)+f_k(y)-F(y) \\ +f_k(x)-f_k(y) \end{array} \right|$$

$$= \Big| \underbrace{F(x)-f_k(x)}_{\text{uniform convergence}} + \underbrace{f_k(x)-f_k(y)}_{\text{continuity}} + \underbrace{f_k(y)-F(y)}_{\text{uniform convergence}} \Big|$$

Apply the triangle inequality.

$$\leq |F(x)-f_k(x)| + |f_k(x)-f_k(y)| + |f_k(y)-F(y)|$$

We bound each term by $\varepsilon/10$. and get:

$$|F(x)-F(y)| < \frac{\varepsilon}{10} + \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon.$$

<u>Proof</u>: Given $\varepsilon > 0$ we pick:

N_F such that:
$$k > N_F \implies |F(x)-f_k(x)| < \frac{\varepsilon}{10}$$

δ_k such that:
$$|x-y| < \delta_k \implies |f_k(x)-f_k(y)| < \frac{\varepsilon}{10}.$$

We obtain, by the observations above,
If $|x-y| < \delta_k$ then $|F(x)-F(y)| < \frac{3 \cdot \varepsilon}{10} < \varepsilon$.