CLTA Talk # 2

From Colourings to Fixed Points

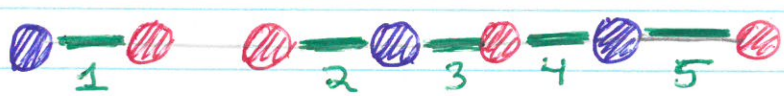
This talk could appear in:

MAT C32 — Graph Theory and Agorithms
MAT C27 — Topology
MAT C44 — Combinatorics.

Consider a string of beads

We require that the first bead is blue
last bead is red.

We call such a string WELL COLOURED.



Observation: In a well coloured string there are an ODD number of segments with differently coloured end points.

Pf: If we switch colour an even number of times then the string will have the same start/end colour.
Thus, it will not be well coloured

(!) Odd implies there is at least one.

Let's use this to do something neat!

Thm: If $f: [0,1] \to [0,1]$ is continuous then there is a FIXED POINT $x_* \in [0,1]$ such that $f(x_*) = x_*$.

Question: How can we get this from first year calculus? IVT?

[Define $g(x) = x - f(x)$. Observe: $g(0) < 0$ and $g(1) > 0$]

Proof: Assume $f: [0,1] \to [0,1]$ is continuous. We call $[0,1] = I_0$

Divide I_0 is to two equal parts:

$x_0 = 0 \quad x_1 = \frac{1}{2} \quad x_1 = 1$

We colour the point x_i

BLUE if $x_i - f(x_i) < 0$

RED if $x_i - f(x_i) > 0$

Question: What about $x_i - f(x_i) = 0$? (If that happens we're done.)

Question: Why is I_0 well coloured?

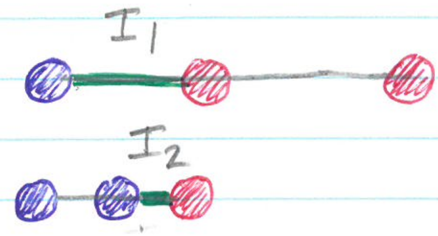
There are an odd number of segments with distinctly coloured edges.

$I_1 = [x_i, x_{i+1}]$ to be the first.

Pf (con't)

Subdivide I_1 in to two equal subintervals. Colour the endpoints.

Obtain : $I_2 \subseteq I_1$



Each I_n has well coloured endpoints. We obtain:

$$\ldots \subseteq I_n \subseteq \ldots \subseteq I_3 \subseteq I_2 \subseteq I_1 \subseteq I_0 .$$

Each I_k is closed and we get : $x_* \in \bigcap_{i=0}^{n} I_n$

Sub-Claim : $f(x_*) = x_*$.

Let $I_k = [x_k^-, x_k^+]$.

We have $\lim_{k\to\infty} x_k^- = \lim_{k\to\infty} x_k^+ = x_*$.

Moreover, $f(x_k^-) > x_k^-$
$f(x_k^+) < x_k^+$

By continuity :
$\lim x_k^- \leq \lim f(x_k^-) = f(x_*)$
$f(x_*) = \lim f(x_k^+) \leq \lim x_k^+$

Thus, $f(x_*) = x_*$.