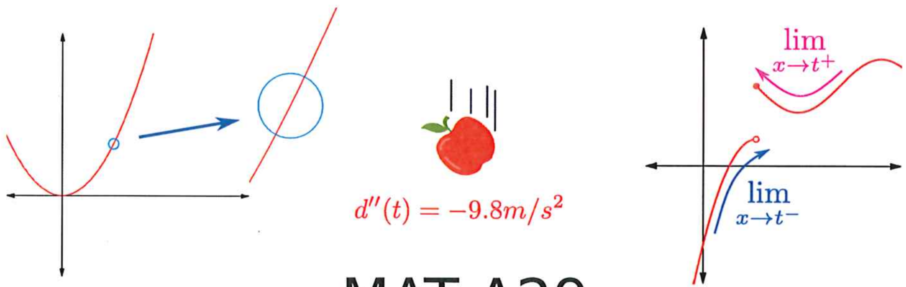
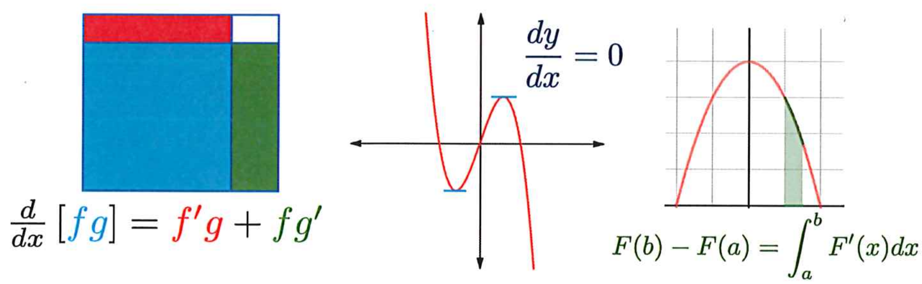
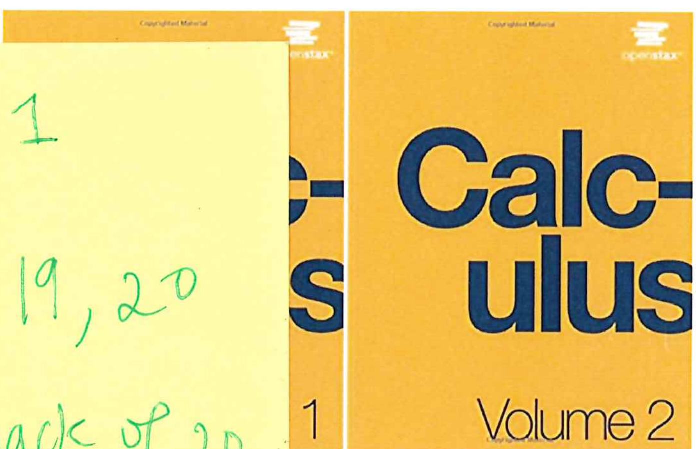
Week 1: Welcome to the Course and Introduction to Functions



MAT A29
Calculus I for the Life Sciences





Week 1

Pages 19, 20

(and back of 20 Not on Quercus)

Grading Scheme

	$10\% + 20\% =$	30%
n	$1 \times 40\% =$	40%
Assignments	$(6-1) \times 3\% =$	15%
Quizzes	$(6-1) \times 3\% =$	15%

Functions: OpenStax §1: Functions and Graphs

Definition: Functions

A **function** consists of a set of inputs, outputs, and a rule for associating inputs to outputs. The valid inputs are the **domain** of the function, and the valid outputs are the **range**.

Example: Various Functions

- Inputs: Students. Outputs: Numbers. Rule: Every student has a student number.
- Inputs: Students. Outputs: Numbers. Rule: Every student has a height.
- Inputs: $\mathbb{R}$, Outputs: $\mathbb{R}$. Rule: To the number x assign x^2.

$\mathbb{R}$ = "real numbers"

Question: A Human Range

Do you think that Height = 10 meters is in the range of students' heights?

No! It is not in the range.
Humans (students) don't grow to be 10 m tall.

Question: A Mathematical Domain

Is -1 in the domain of the function $f(x) = x^2$?

Yes! $f(-1) = (-1)^2 = 1$.

Therefore, -1 is a valid input of $f(x)$.

So, -1 is in the domain of $f(x)$.

Question: A Mathematical Range

Is -1 in the range of the function $f(x) = x^2$?

No! -1 is not in the range.

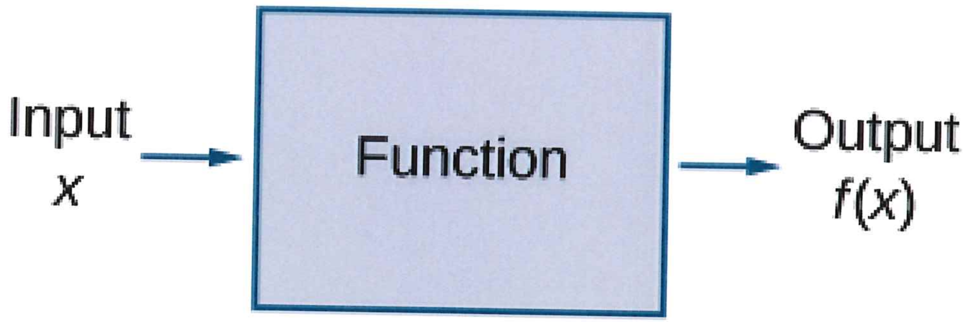
Why not? We know $x^2 \geq 0$

and so $x^2 \neq -1$ for any x.

Remark: Ways of Thinking of Things

In mathematics, it is often helpful to have multiple ways of thinking about the same thing. Some ways of thinking about things work better for some people.

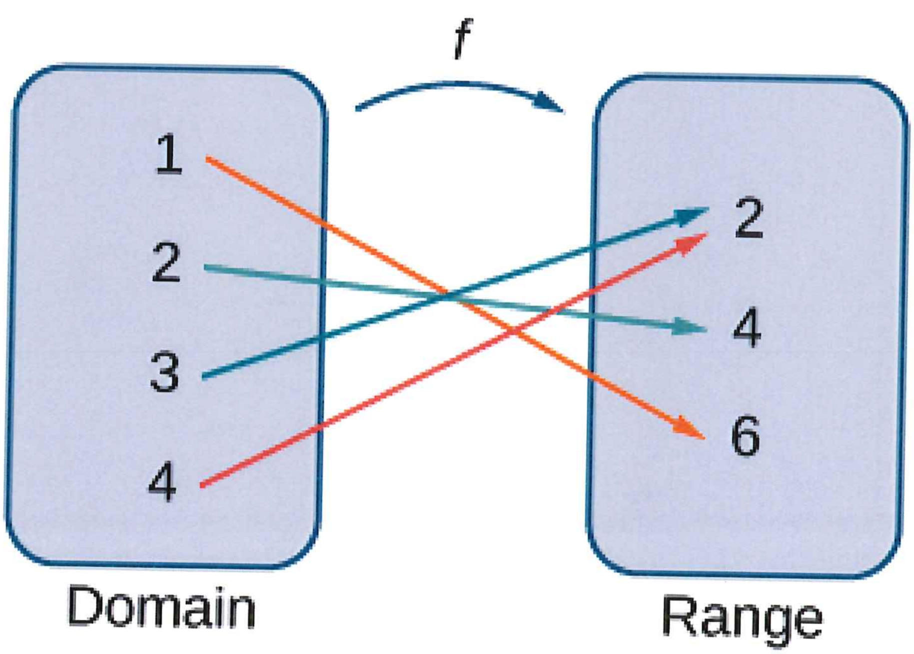
Functions as Black Boxes



$-1 \longrightarrow \boxed{x^2} \longrightarrow 1$

OpenStax §1.1 Figure 1.2

Functions as Mappings

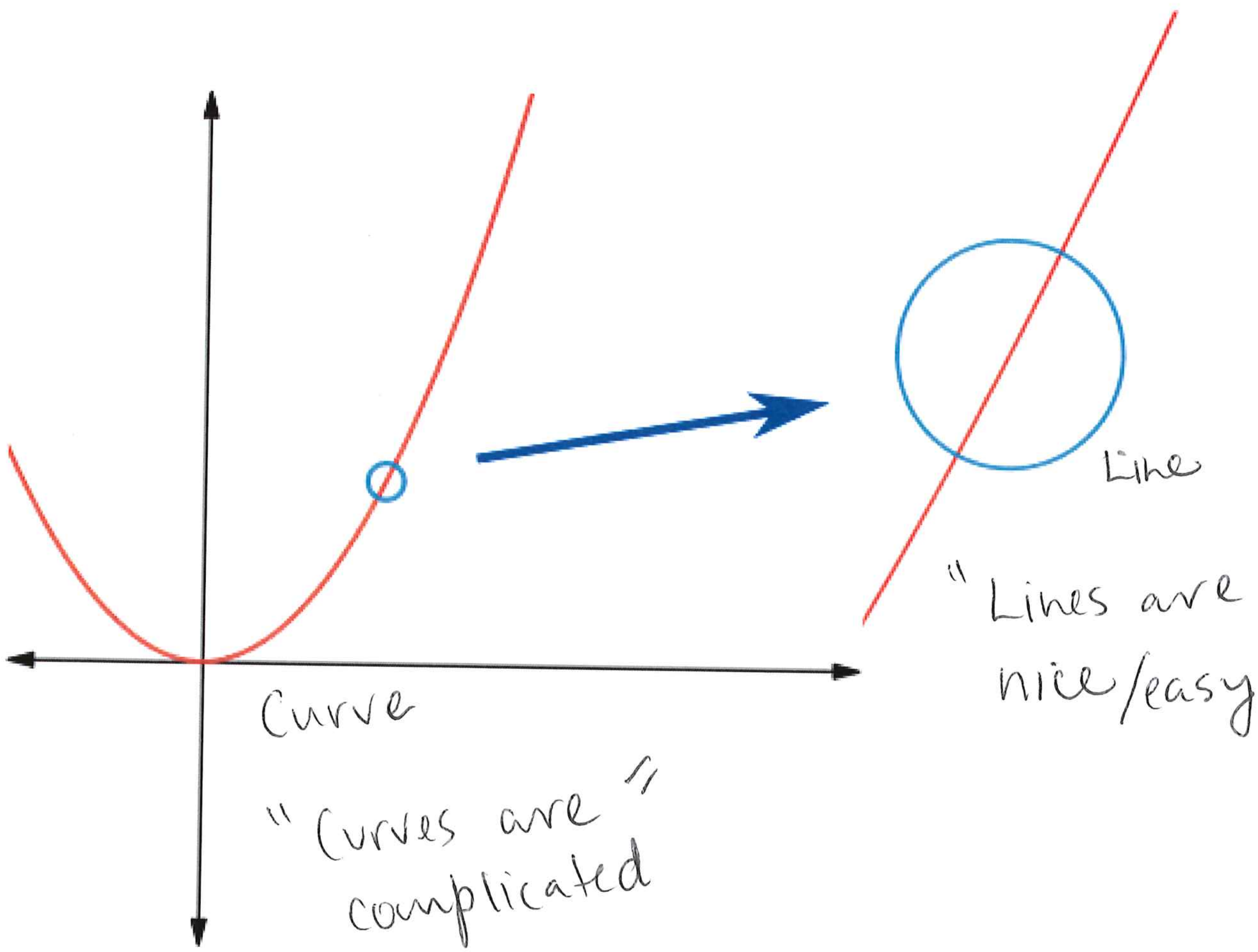


We can also present this data as a **table of values**:

x	1	2	3	4
$f(x)$	6	4	2	2

Example: A Preview of Calculus

If we zoom in very close on any "nice" graph, we get a line.



Activity: Try This in Desmos (5 min)

Open up Desmos, and type in a function. Zoom in on any point, and it'll eventually become a line.

Question: "Nice" Functions

What do we mean by a nice function? What are the ingredients of such a function?

- polynomials - sin, cos, tan - $\sqrt{\ }$ - x^k
- e^x and $\log(x)$

For example $\frac{1}{1-x^2}$ or $\ln(1-e^{\sin(x)})$

To find more examples or explanation.

Definition: Lines: OpenStax §1.2: Basic Classes of Functions

A **linear function** is $y = mx + b$ where m and b are some numbers.
The **x-intercept** is where $y = 0$ and the **y-intercept** is where $x = 0$.

m = slope
b = y-intercept.

Question: Intercepts

Which of the following lines has x-intercept $x = 2$ and y-intercept $y = 4$?

1. $y = 2x + 4$ $\Rightarrow y = 4$ $\quad\Rightarrow$ means "implies"
2. $y = 4x - 2$ $\Rightarrow y = -2$
3. $y = 4x + 2$ $\Rightarrow y = 2$
4. $y = -2x + 4$ $\Rightarrow y = 4$

To find y-intercept: plug in $x=0$.
This eliminates (2) and (3) as possibilities.

To find x-intercept: solve for $y=0$.

(1) $y=0$

$$2x+4=0$$
$$\Rightarrow 2x = -4$$
$$\Rightarrow x = \frac{-4}{2} = -2$$

The x-intercept is $x = -2$

(4) $y=0$

$$-2x+4=0$$
$$\Rightarrow -2x = -4$$
$$\Rightarrow x = \frac{-4}{-2} = +2.$$

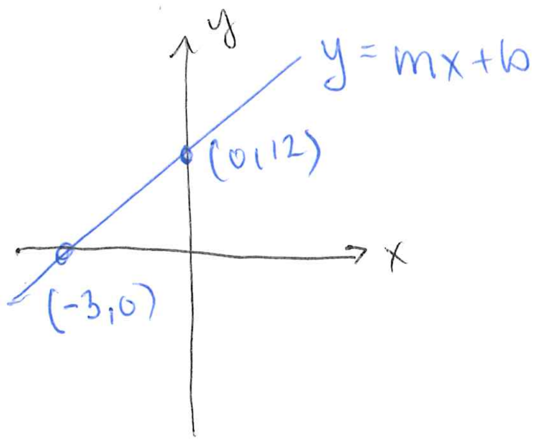
The x-intercept is $x = 2$

Activity: Micro-Assignment: 5 min

Find an equation of the form $y = mx + b$ with x-intercept $x = -3$ and y-intercept $y = 12$.

Solve this and hand-in on top of the piano.

- Anonymous.
- No need to copy the question.



We know $b = 12$.

We calculate m:

$$m = \text{“}\frac{\text{rise}}{\text{run}}\text{”} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$= \frac{12 - 0}{\cancel{-3-0}\ \ 0-(-3)} = \frac{12}{\cancel{-3}\ \ 3} = +4$$

We get: $y = mx + b = 4x + 12$.

Break until 14:15

Definition: Lines: Slope — OS §1.2 Eq 1.3

The **slope** of a line L through points (x_0, y_0) and (x_1, y_1) is:

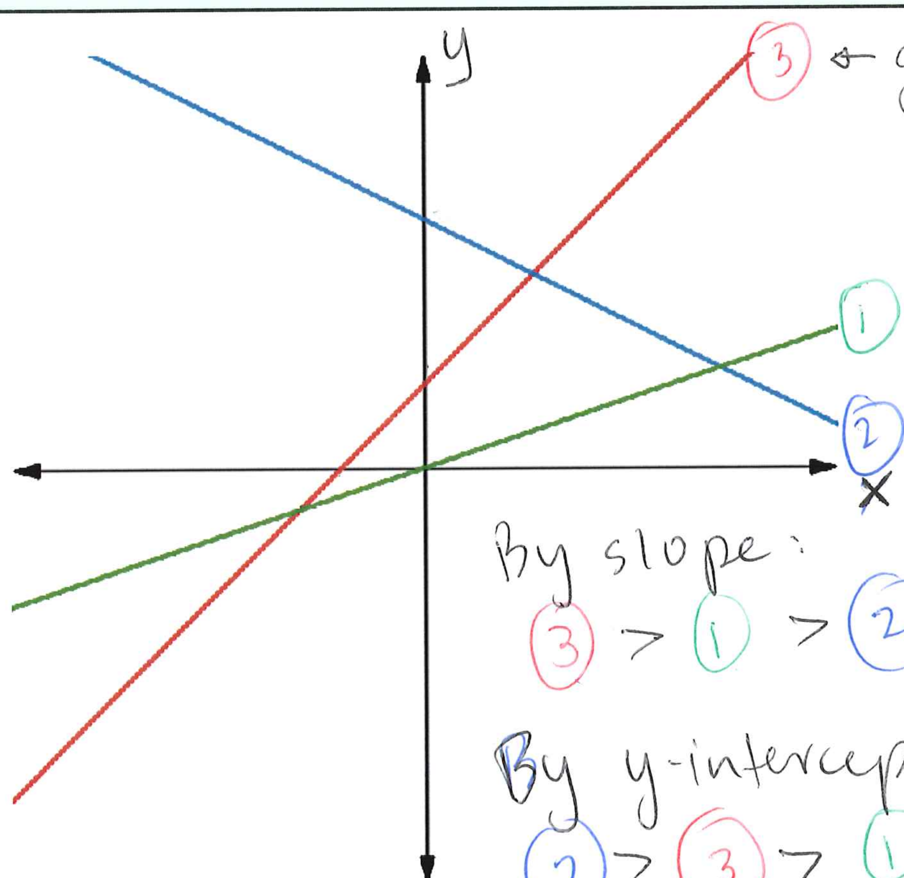
$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$

Δ = Delta = Difference = Change

where: $\Delta y = y_0 - y_1 =$ "rise" and $\Delta x = x_0 - x_1 =$ "run".

Activity: Discuss: 3 min

Which line has the greatest slope? Which line has the smallest y-intercept?
Sort the lines according to (i) their slope, and (ii) their y-intercept.
Which lines are increasing? Which are decreasing?



← greatest slope.

← Smallest y-intercept

By slope:
3 > 1 > 2

By y-intercept:
2 > 3 > 1

Increasing: 3 and 1
Decreasing: 2

Question: Find A Line

Find the equation of a line $y = mx + b$ passing through the point $(2, 9)$ with slope $m = -3$.

We need to find the value of b.

We have: $9 = (-3) \cdot 2 + b$

$\Rightarrow 9 = -6 + b$

$\Rightarrow 15 = b$

Therefore, the line is $y = -3x + 15$.

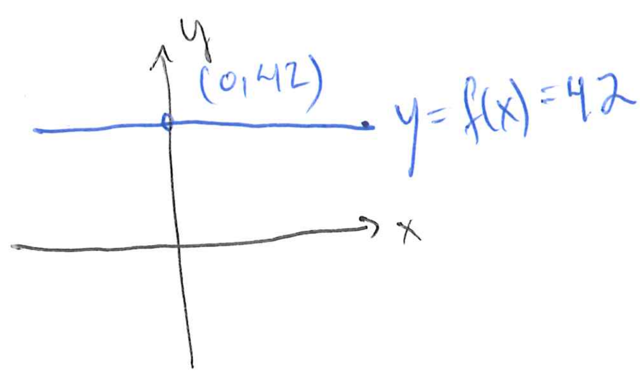
Domain = Valid inputs Range = Valid outputs.

Question: A Funny Domain And Range

What are the domain and range of the line $f(x) = 42$?

The range is $R = \{42\}$ ← The set containing 42.

The domain is $D = \mathbb{R}$ because all inputs are valid.



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Activity: Class Discussion: 2 min

What are some other possible domains and ranges of lines?

Consider the line $x = y$.

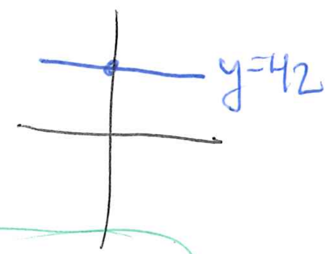
For this line: Domain $= (-\infty, \infty) = \mathbb{R}$

Range $= (-\infty, \infty) = \mathbb{R}$.

Consider $y = 42$

For this line: Domain $= \mathbb{R}$

Range $= \{42\}$



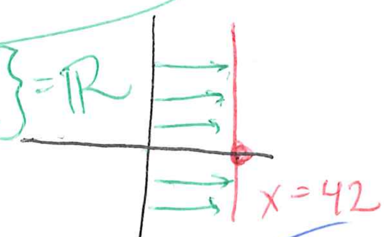
Switch x and y axes

$f(y) = 42$

Consider $x = 42$

For this line: Domain $= \{y \in \mathbb{R}\} = \mathbb{R}$

Range $= \{42\}$



Cool Idea:

You could have Domain = Integers.

If we plug in integers, we get a lot of "gaps".

Definition: Intervals OpenStax §1.1

An **interval** is a collection of numbers defined inequalities: $\leq$, $\geq$, $<$, $>$.
For example,

- $\{x : 1 < x < 2\} = (1, 2)$
- $\{x : \pi \leq x < 4\} = [\pi, 4)$
- $\{x : x < 1\} = (-\infty, 1)$
- $\{x : -3 \leq x\} = [-3, \infty)$

We use [brackets for $\leq$ inequalities, and (brackets for $<$ inequalities.
Similarly, we use] brackets for $\geq$ inequalities, and) brackets for $>$ inequalities.

Definition: The Real Numbers

The <u>real numbers</u> are $(-\infty, \infty) = \mathbb{R}$.

"reals" $= \mathbb{R}$

Question: Find the Interval

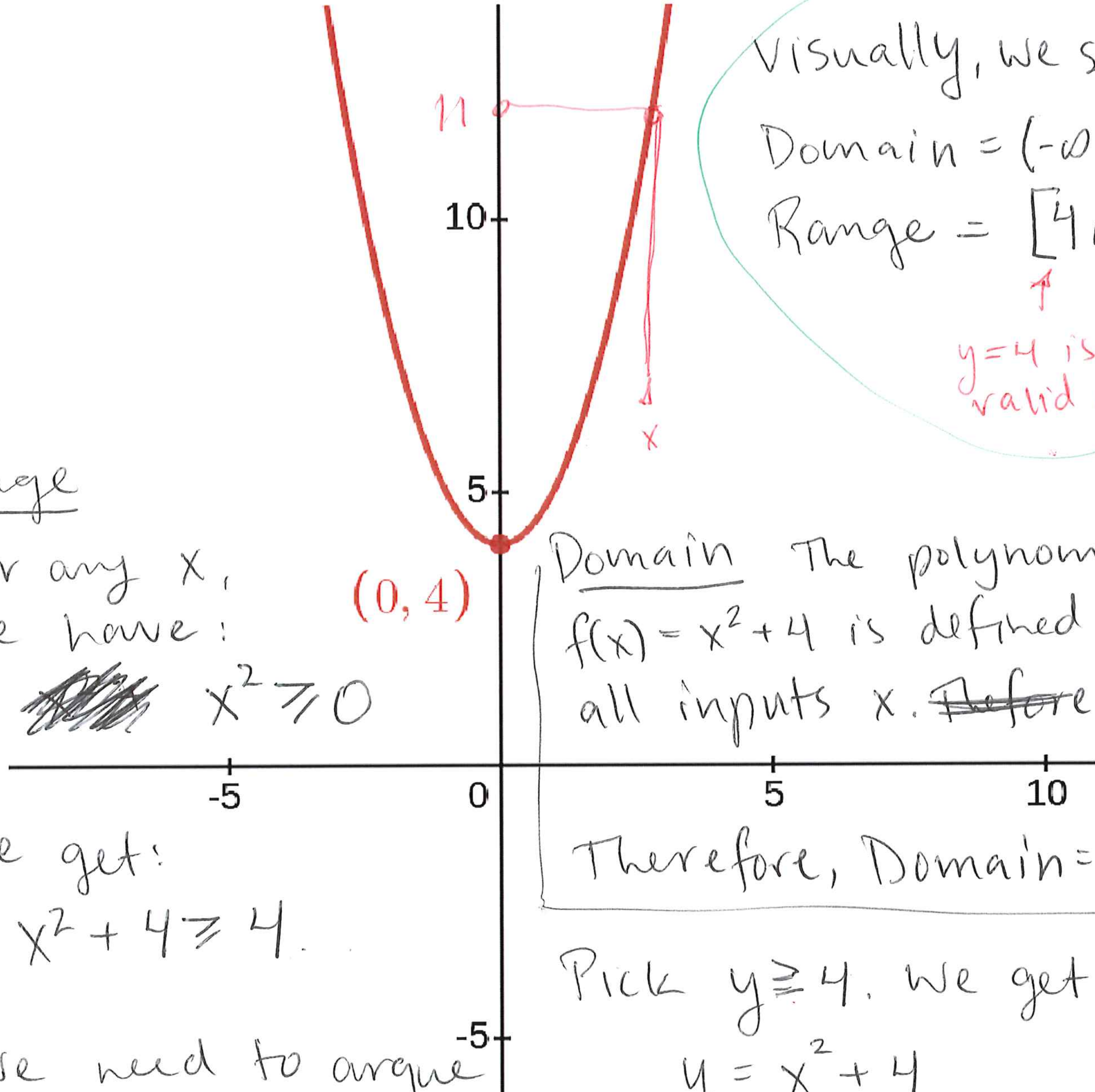
Which of the following intervals represents: $\{x : x \leq -2\}$?

1. $(-2, \infty)$
2. $(-\infty, -2)$
3. $(-\infty, -2]$
4. $[-2, \infty)$

$(-\infty, -2] = \{x : x \leq -2\}$.

Question: Find the Domain and Range

What are the domain and range of $f(x) = x^2 + 4$?



Visually, we see:

Domain $= (-\infty, \infty)$

Range $= [4, \infty)$

$y = 4$ is a valid output.

Domain The polynomial $f(x) = x^2 + 4$ is defined for all inputs x. ~~Therefore~~

Therefore, Domain $= \{x \in \mathbb{R}\}$.

Range

For any x, we have:

$x^2 \geq 0$

We get:

$x^2 + 4 \geq 4$.

We need to argue that every $y \geq 4$ actually occurs.

Pick $y \geq 4$. We get:

$y = x^2 + 4$

$\Rightarrow y - 4 = x^2$

$\Rightarrow \pm\sqrt{y-4} = x$

For this x, we have:

$f(x) = y$.

Therefore, y is in range.

Remark: Writing matters.

In this class, I will heavily emphasize writing out full solutions. This is (probably) quite different from your experience in highschool where only the final answer is graded. We're doing this because written communication is a highly valuable skill. (Everyone wants good communication!)

Let's write out a full solution for: "Find the domain and range of $f(x) = x^2 + 4$."

- Domain:

For any value of x, we can compute x^2+4 and obtain a valid result. Therefore, the domain is $(-\infty, \infty)$.

- Range:

Notice: $x^2+4 \geq 0+4 = 4$. Therefore, the range is contained in $[4,\infty)$. Pick a value $y \in [4,\infty)$. We solve:

$$y = x^2+4 \Rightarrow x = \pm\sqrt{y-4}$$

Notice: $y \geq 4$ and so the square root is defined. We get: $f(x) = y$ and so all possible $y \in [4,\infty)$ occur and the range is $[4,\infty)$.

https://www.reddit.com/r/UTSC/comments/1fpdfpl/how_much_work_do_i_need_to_show_for_a_single_math/

Definition: Composition

Given two functions $f(x)$ and $g(x)$ one can **compose** them:

$$f(g(x)) = (f \circ g)(x) = \text{“f of g of x”} \qquad g(f(x)) = (g \circ f)(x)$$

This is operation is very important later in the course when we study the chain rule.

Question: Two Simple Compositions

Write out the composition $(f \circ g)(x)$ and $(g \circ f)(x)$ for $f(x) = \sqrt{x}$ and $g(x) = x + 9$.

$$(f \circ g)(x) = f(g(x)) = f(x+9) = \sqrt{x+9}$$

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 9$$

(!) Notice that $f \circ g$ and $g \circ f$ are almost never equal. In this example:

$$(g \circ f)(1) = \sqrt{1} + 9 = 1 + 9 = 10$$

$$(f \circ g)(1) = \sqrt{1+9} = \sqrt{10} \approx 3$$

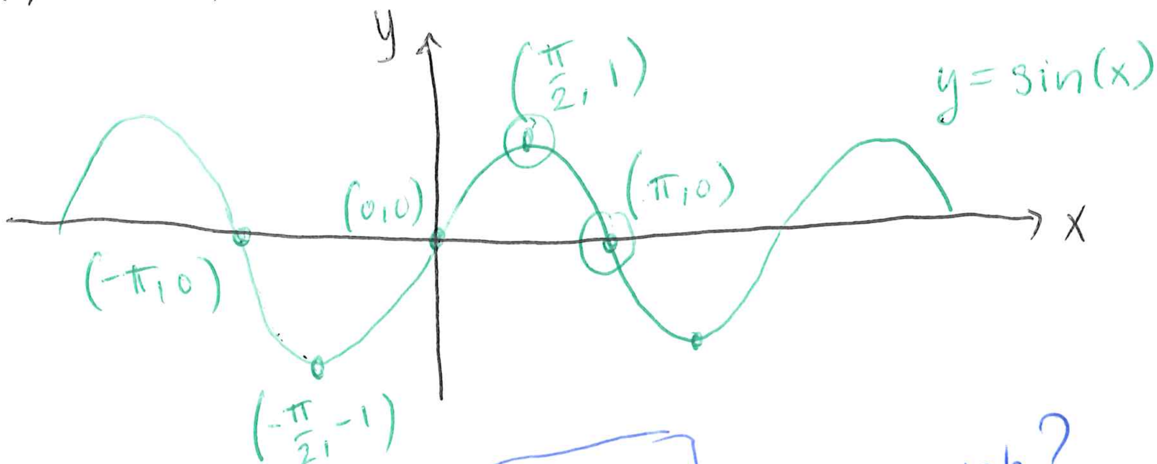
multiplication is a different operation

$$f(x)g(x) = \sqrt{x}(x+9) = x^{\frac{3}{2}} + 9x^{\frac{1}{2}}$$

Example: Composition and Transformation

How does the composition $g(x) = \sin(2x)$ modify the graph of $y = \sin(x)$?
Highlight any key points in both graphs.

The composition here is: $g(x) = \sin(2x) = f(h(x))$ for $f(x) = \sin(x)$ and $h(x) = 2x$.

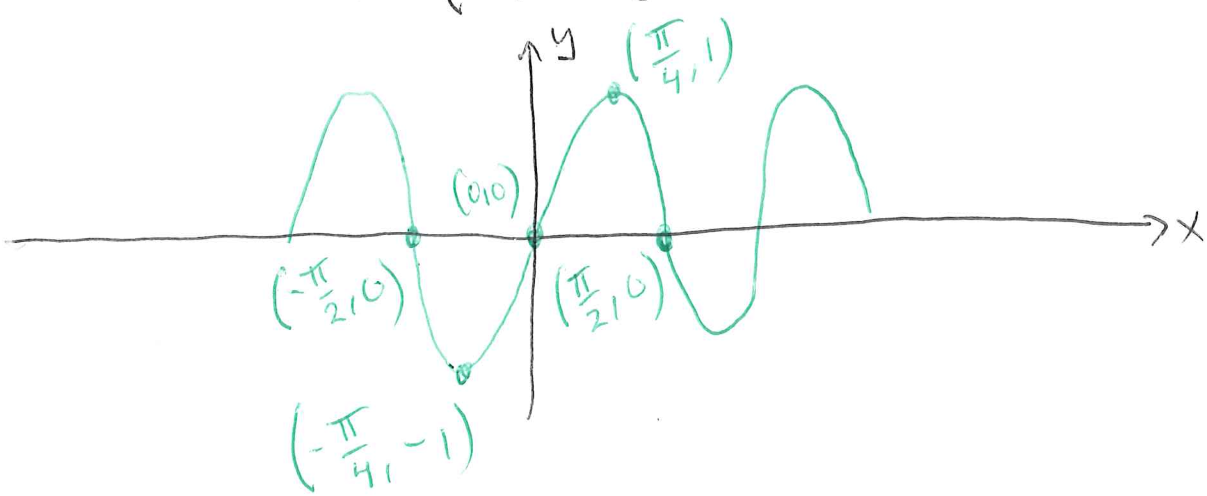


? Does it stretch or $\boxed{\text{squish}}$ the graph?

We know $\sin\left(\frac{\pi}{2}\right) = 1. \iff \sin\left(2\left(\frac{\pi}{2\cdot 2}\right)\right) = 1$

$\iff \sin\left(2\left(\frac{\pi}{4}\right)\right) = 1$

$\sin(\pi) = 0 \iff \sin\left(2\left(\frac{\pi}{2}\right)\right) = 0$



Do we need to modify y-values?
Yes — For something like $y = 2\sin(x)$

Activity: Micro-Assignment: (5 min)

What are the domain and range of $f(x) = \sqrt{x-6}$?

- Anonymous
- Front of room.

Domain:

For $\sqrt{x-6}$ to be defined we need $x-6 \geq 0 \iff x \geq 6$.
Therefore, the domain is $[6, \infty)$.

Range

The square root only outputs numbers in $[0, \infty)$.

We pick $y \in [0, \infty)$ and solve for x:

$y = \sqrt{x-6} \iff y^2 = x-6 \iff x = y^2+6$.

This x is in the domain because $x \geq 0^2+6=6$.
~~Refor~~. Therefore $f(x)=y$ and the range is $[0, \infty)$.

What's up with $\pm$?

$x^2 = 4 \implies \sqrt{x^2} = \sqrt{4} = 2 \implies |x| = 2 \implies x = \pm 2$

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The $\pm$ occur when we solve $x^2 =$ something.

Question: Domain of a Rational Function

What is the **domain** of $f(x) = \frac{1}{x-3}$?

Notice: If $x - 3 = 0$ then the function is undefined.

This means the domain is: $\{x \in \mathbb{R} : x \neq 3\}$.

$= (-\infty, 3) \cup (3, \infty)$.

"cup" or "union"

$X \cup Y$ is all the stuff in X or Y.

$= \{x \in \mathbb{R} : x < 3\} \cup \{x \in \mathbb{R} : x > 3\}$

$\{x \in \mathbb{R} : x \neq 3\}$.

Question: Range of a Rational Function

What is the **range** of $f(x) = \frac{1}{x-3}$?

A division will never output zero.

We think the range is $(-\infty, 0) \cup (0, \infty)$.

Pick any $y \neq 0$ if $y = \frac{1}{x-3}$ then $y(x-3) = 1$

This gives: $yx - 3y = 1 \Rightarrow yx = 1 + 3y \Rightarrow x = \frac{1+3y}{y}$.

For this value of x we have:

$$\frac{1}{x-3} = \frac{1}{\left(\frac{1+3y}{y}\right) - 3} = \frac{1}{\frac{1}{y} + 3 - 3} = \frac{1}{\frac{1}{y}} = y$$

Therefore range is $(-\infty, 0) \cup (0, \infty)$.

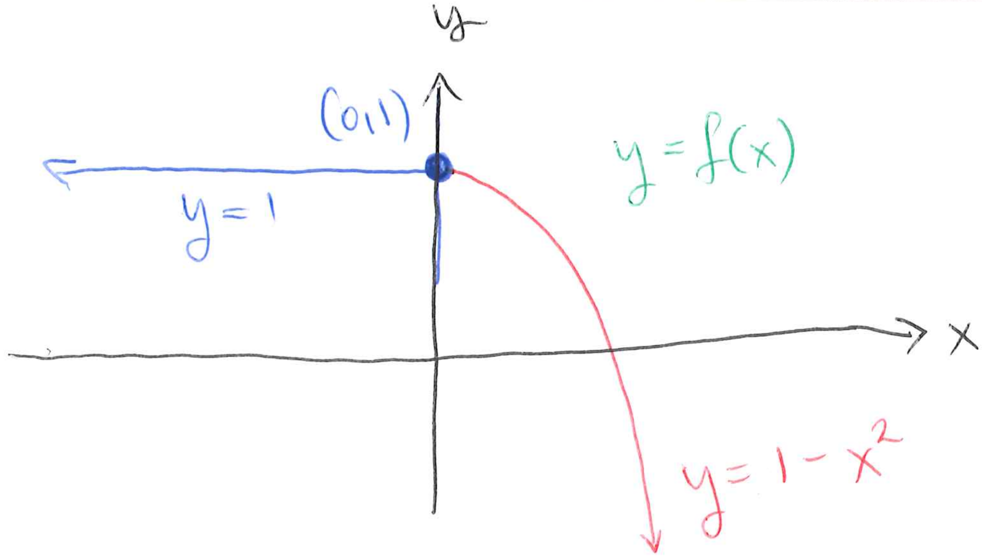
Definition: Piecewise Functions

A **piecewise** function has different equations for different parts of its domain.

Question: Sketch A Function

Sketch the function:

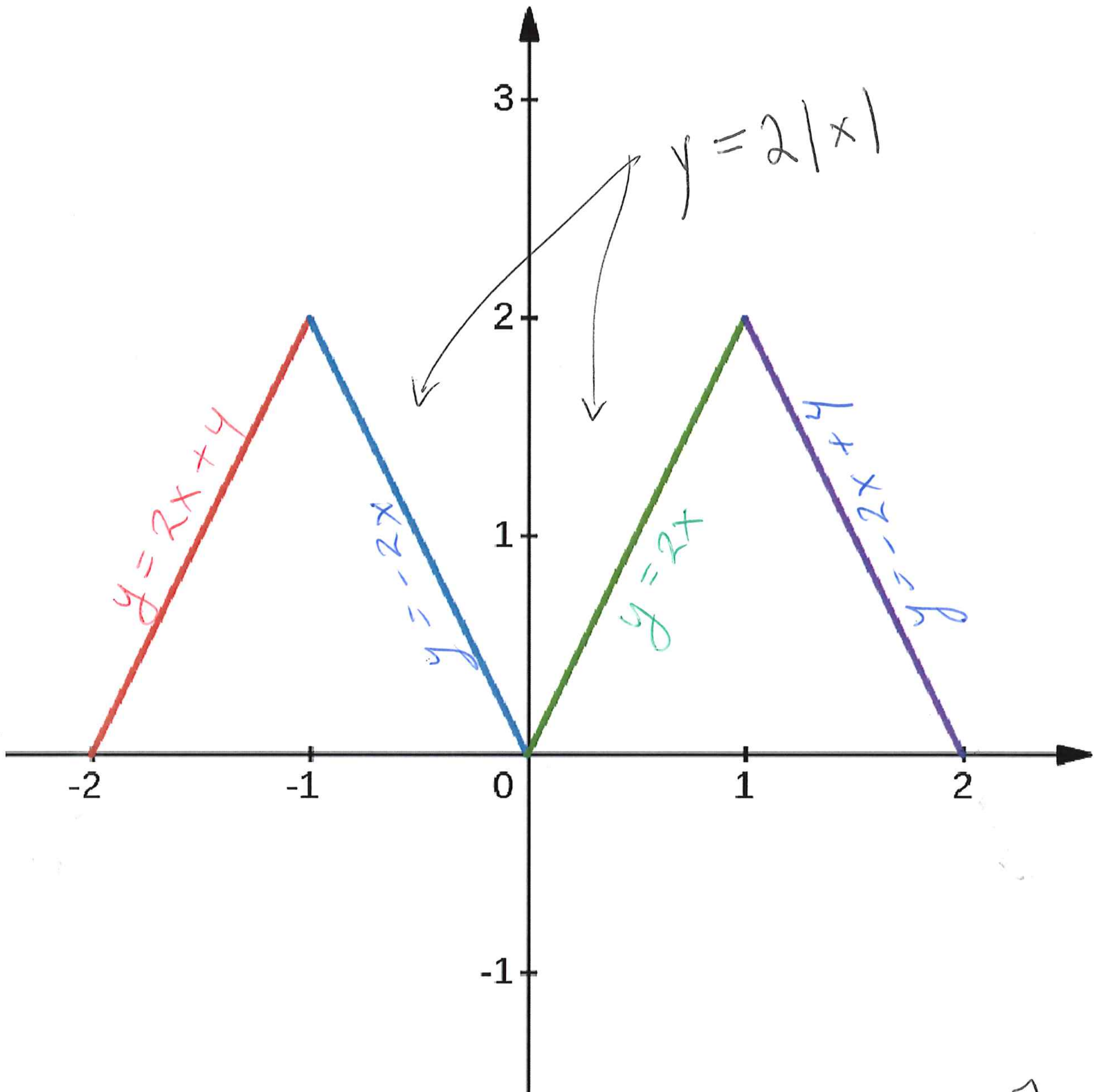
$$f(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x^2 & x > 0 \end{cases}$$



= the point (0,1) is on the graph

Activity: Class Discussion: 3 min

What can you say about this piecewise graph using the material we learned this week?



Domain = [-2, 2]

Range = [0, 2]

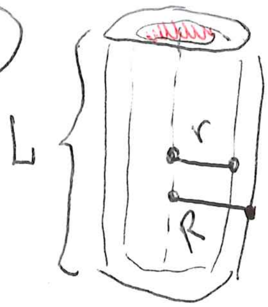
Example: An Application: Femur Strength

A femur (or thighbone) is a bone in your leg. It is essentially a long thin tube filled with marrow. It has outer radius R and inner radius r where $\boxed{R > r > 0}$. The density of bone is approximately 1.8g/cm^3 and marrow is approximately 1g/cm^3. A medically significant ratio of a bone is: $k = R/r$. Suppose that a femur has length $L > 0$ in centimeters. (Adapted from Starr, Cecie, A. Evers Christine, and Starr Lisa. *Biology: concepts & applications.* Cengage Learning, 2018.)

1. Sketch a diagram of a femur (as a long thin tube) and label it with R, r, and L.
2. What are the possible values of k?
3. Express the mass $m(k)$ of the femur as a function of k.
4. What is the domain of $m(k)$? What is the range $m(k)$?
5. The following question is not a math question, it requires some thinking about biology: If an elderly person falls, do they want $k \approx 1$ or $k \approx 10$?

(5) If $k \approx 1$ then $r \approx R$ ad the femur is thin.
If the femur walls are thin, they break easily.
An elderly person wants $k \approx 10$ NOT $k \approx 1$.

(1)



(2) We have $R > r > 0$.
This gives:

$$\frac{R}{r} > \frac{r}{r} > \frac{0}{r} \Leftrightarrow k = \frac{R}{r} > 1$$

(3)

$$\begin{aligned}
\text{Mass} &= (\text{Mass of marrow}) + (\text{Mass of bone}) \\
&= \left(1\frac{g}{cm^3}\right)(\text{vol marrow}) + \left(1.8\frac{g}{cm^3}\right)(\text{vol bone}) \\
&= (1)(L\pi r^2) + (1.8)(L\pi R^2 - L\pi r^2) \\
&= L\pi r^2 + (1.8)L\pi(R^2 - r^2)
\end{aligned}$$

(4) Domain $= \{k > 1\}$
Range $= \{m > L\pi r^2\}$

See Reverse.

$$m(k) = L\pi r^2\left[1 + (1.8)\left(\frac{R^2}{r^2} - 1\right)\right] = L\pi r^2\left[1 + (1.8)(k^2 - 1)\right]$$

The mass function is:

$$m(K) = L\pi r^2\left[1 + (1.8)(K^2 - 1)\right].$$

Notice that $K > 1$ according to part 2.

Therefore the domain of m is $\{K > 1\} = (1, \infty)$.

The range is determined by the smallest possible value of $m(K)$. We have:

$$K^2 - 1 > 0$$

and so:

multiplication

$$m(K) = L\pi r^2\left[1 + (1.8)(K^2 - 1)\right]$$
$$> L\pi r^2\left[1 + (1.8)\cdot 0\right] = L\pi r^2.$$

The range is therefore $(L\pi r^2, \infty)$.