

Week 2: Polynomials

Definition: Polynomials

OS §1.2 Eq. 1.7

A polynomial is a function of the form:

leading
coeff.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

The degree d of a polynomial is the highest power in it. $= n$ The number in front of the highest power is the leading coefficient. $= a_n$

Example: Coefficients

Write out the coefficients of:

$$f(x) = 2x^3 + 9x^2 = 2x^3 + 9x^2 + 0x + 0.$$

$a_3 = 2 \quad a_2 = 9 \quad a_1 = 0 \quad a_0 = 0.$

Definition: End Behaviour of Polynomials

The end behaviour of $f(x)$ is what $f(x)$ does when x gets very large ($x \rightarrow \infty$). or ($x \rightarrow -\infty$)The end behaviour of $f(x) = x^1$:

$$f(x) \rightarrow \infty \text{ as } x \rightarrow \infty$$

$$\text{and } f(x) \rightarrow -\infty \text{ as } x \rightarrow -\infty.$$

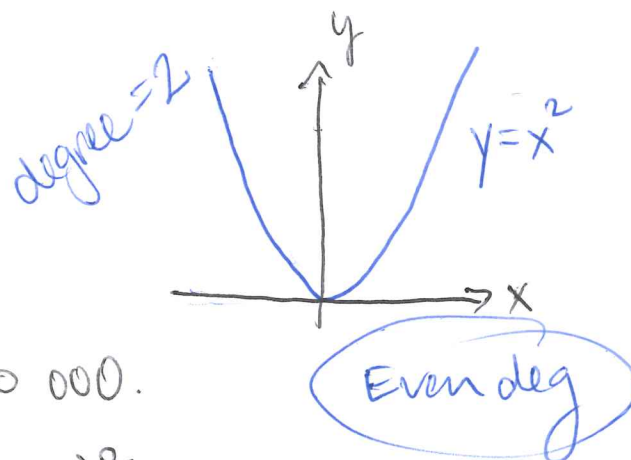
" $f(x)$ goes to infinity as x goes to infinity"

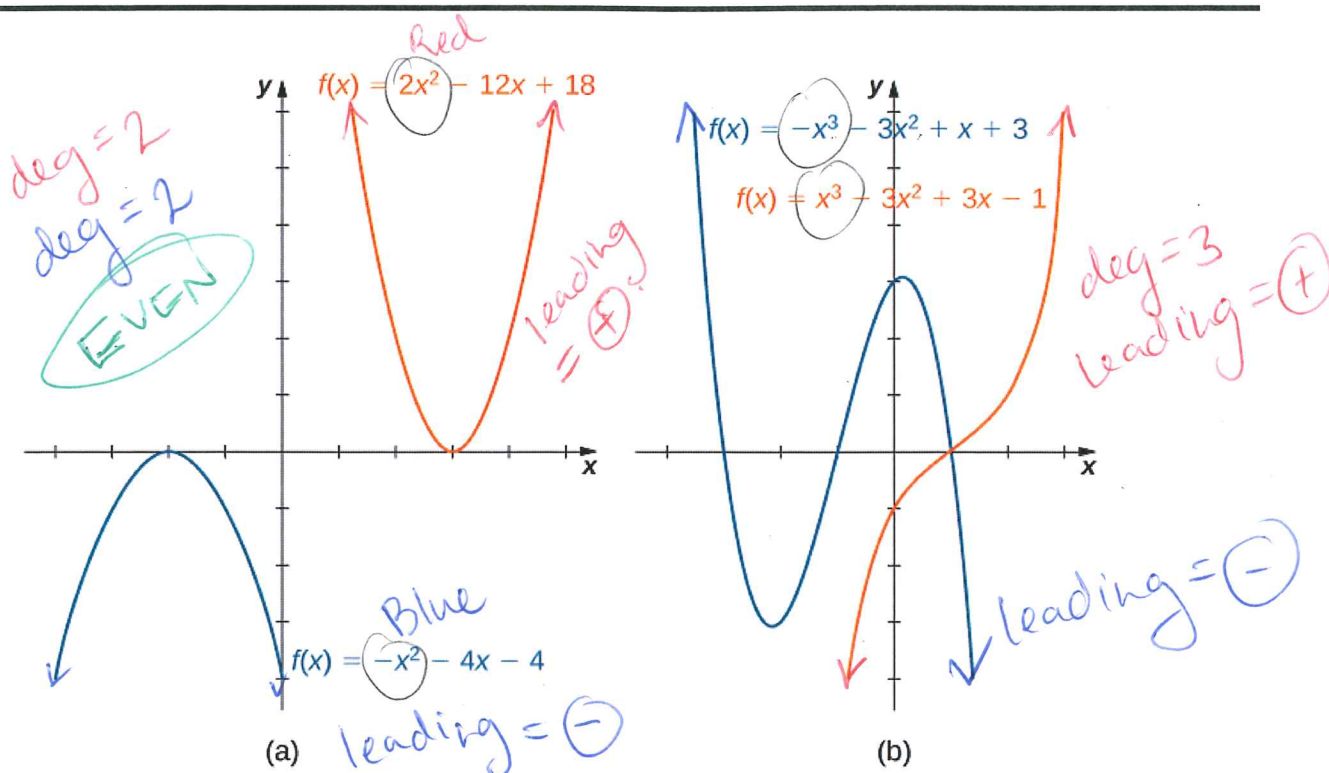
The end behaviour of $f(x) = x^2$:

$$f(x) \rightarrow \infty \text{ as } x \rightarrow \infty$$

$$f(x) \rightarrow \infty \text{ as } x \rightarrow -\infty$$

For example $f(-1000) = 1000000.$
 $= (-1000)^2$





OpenStax §1.2 Figure 1.19

Example: End Behaviour Visually

Describe the end behaviour of the functions above.

AND

- (a) $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$.
 $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.
 These show we can't figure out end behaviour using JUST degree. We need to consider leading coeff.

- (b) $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.
 $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$.

The end behaviour of a polynomial is COMPLETELY determined by

even/odd of degree and (+)/(-) of leading coeff.

Question: Describe End Behaviour

Using the words "DEGREE d " and "LEADING COEFFICIENT a " make a short paragraph which describes the end behaviour of polynomials of even degree.

If a polynomial has even degree, then ~~the end behaviour is the same~~
DEGREE d is even.

If the LEADING COEFFICIENT $a > 0$ then
 $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$.

If the LEADING COEFFICIENT $a < 0$ then
 $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

**Activity: Micro-Assignment: (5 min)**

Use the words

- DEGREE d
- LEADING COEFFICIENT a
- ~~THE~~ $x \rightarrow \infty$ ~~END~~
- ~~THE~~ $x \rightarrow -\infty$ ~~END~~

to make a short paragraph explaining the end behaviour of polynomials of odd degree.

$$-f(x) = f(-x)$$

Definition: Symmetry

A graph has **even symmetry** if $f(x) = f(-x)$. It has **odd symmetry** if $f(x) = -f(-x)$.

Example: Even and Odd

The functions $f(x) = x^2, x^4, x^6$ have even symmetry.

The functions $f(x) = x, x^3, x^5$ have odd symmetry.

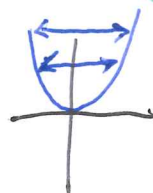
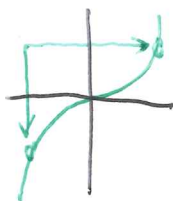
$$f(x) = x^2 \Rightarrow f(x) = f(-x) = (-x)^2 = x^2$$

$$f(x) = x^3 \Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$$

Activity: Discuss: 1 min

Why are these called even and odd symmetries?

- odd symmetry around origin
- even symmetry about y-axis.



- even and odd of degree determines symm.

① even degree does not imply even symm.

We need: all terms in polynomial have even exponents. $\Rightarrow$ even symm

We need: all terms in poly. have odd exponents. $\Rightarrow$ odd symm

Activity: Make Some Examples: (5 min)

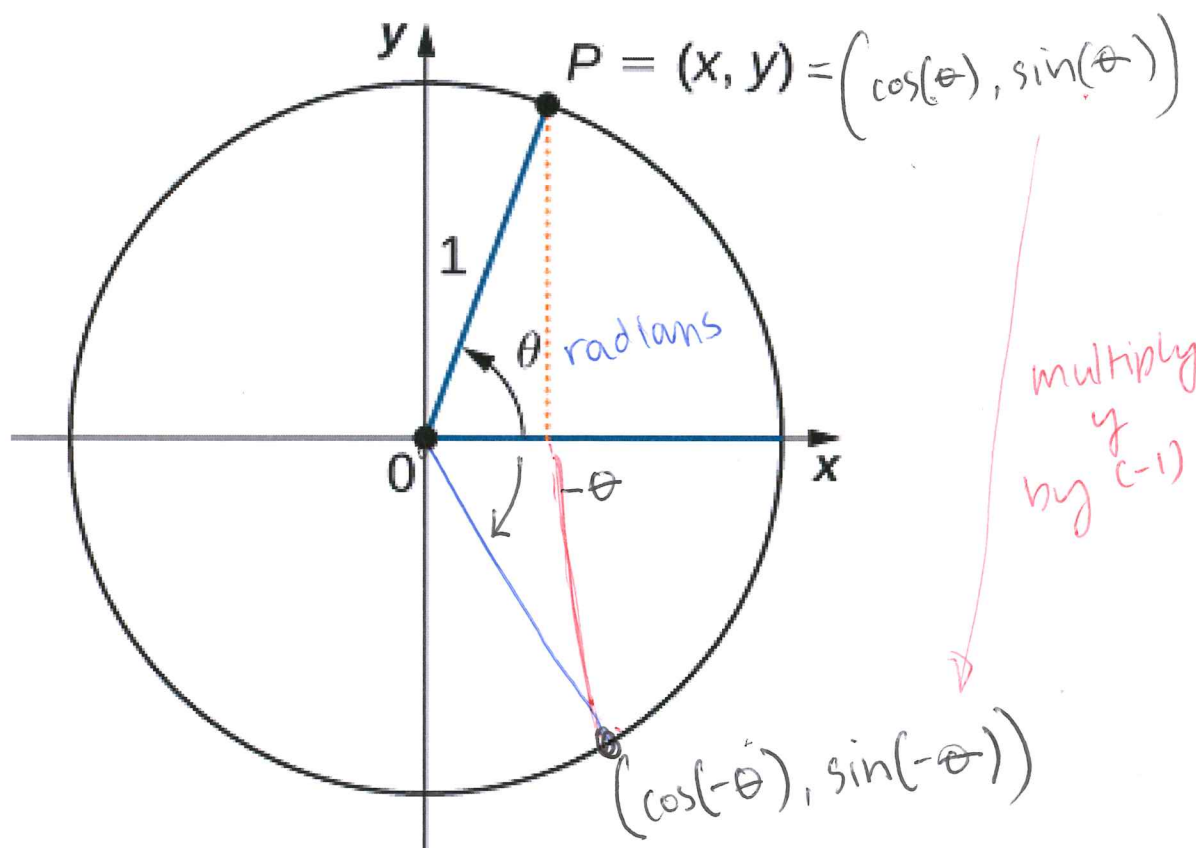
Whenever we meet two similar mathematical properties, it is important to investigate how they are related by finding examples. Find an example of $f(x)$ for each box.

		Even	
		Yes	No
Odd	Yes		
	No		

Definition: The Trigonometric Functions

On the unit circle $x^2 + y^2 = 1$ we have:

$$(x, y) = (\cos(\theta), \sin(\theta)) = (\cos(-\theta), -\sin(-\theta))$$



This gives: $\cos(\theta) = \cos(-\theta)$ and
 $\sin(\theta) = -\sin(-\theta)$.

OpenStax §1.3 Fig 1.31

Therefore: $\cos(\theta)$ is even and
 $\sin(\theta)$ is odd.

Ex: Is $f(x) = x^2 + 1$ even or odd? $\rightarrow f(x) = x^2 + 1 \cdot x^0$

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$$

Therefore, $f(x)$ is even.

Notice:

The exponents 2 and 0 are both even.

		Even	
		Yes	No
Odd	Yes	0	x^3
	No.	x^2	$x^3 + x^2$ $x + 1$ $x^2 + x$

Question: Working Backwards

Write the function $f(x) = \frac{1}{x^2 + 3}$ as a composition $(a \circ b \circ c)(x)$ for three functions: $a(x), b(x), c(x)$.

$$f(x) = \frac{1}{x^2 + 3} = \frac{1}{(((x)^2) + 3)} = \frac{1}{\underline{\underline{(((x)^2) + 3)}}}$$

$$c(x) = x^2$$

$$b(x) = x + 3$$

$$a(x) = \frac{1}{x}$$

For this composition:

$$(a \circ b \circ c)(x)$$

$$= a(b(c(x)))$$

$$= a(b(x^2))$$

$$= a(x^2 + 3)$$

$$= \frac{1}{x^2 + 3}$$

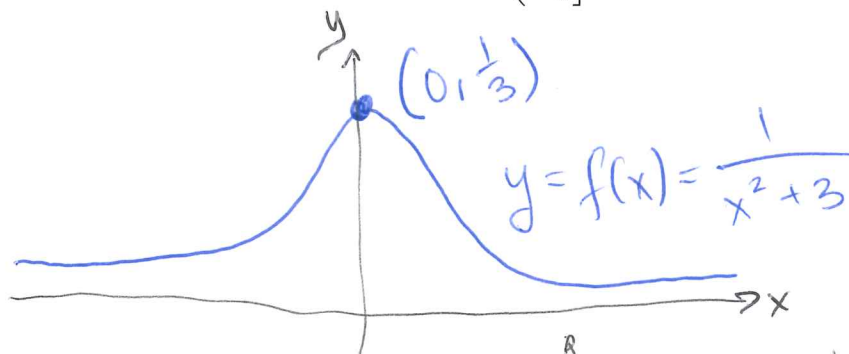


Yay!
It
matches.

Question: A Complicated Range

Find the range of $f(x) = \frac{1}{x^2 + 3}$.

Graph it in Desmos. Notice that the graph is contained in $R = \left(0, \frac{1}{3}\right]$. How do we explain this?



The graph never touches x-axis.

The range of $c(x)$ is $[0, \infty)$. # $c(x) = x^2$

The range of $b(c(x))$ is $[3, \infty)$ # $b(c(x)) = x^2 + 3$

The range of $a(b(c(x)))$ is " $\left(\frac{1}{\infty}, \frac{1}{3}\right]$ "

$a(b(c)) = \frac{1}{x^2 + 3}$

⚠ This notation $\frac{1}{\infty}$ is BAD.
Don't treat infinity as a number.

The range of $a(b(c(x)))$ is $\left(0, \frac{1}{3}\right]$

because $a(b(c(0))) = \frac{1}{3}$ and the end-behaviour of $a(b(c(x)))$ is $\rightarrow 0$.

Why is " $\frac{1}{\infty} = 0$ "? For any large finite number N we have $\frac{1}{N}$ is small.

Remark: An Important Observation

Look at the following pair of solutions to the problem.

Find the domain of $f(x) = \frac{3}{x^2 + 4}$.

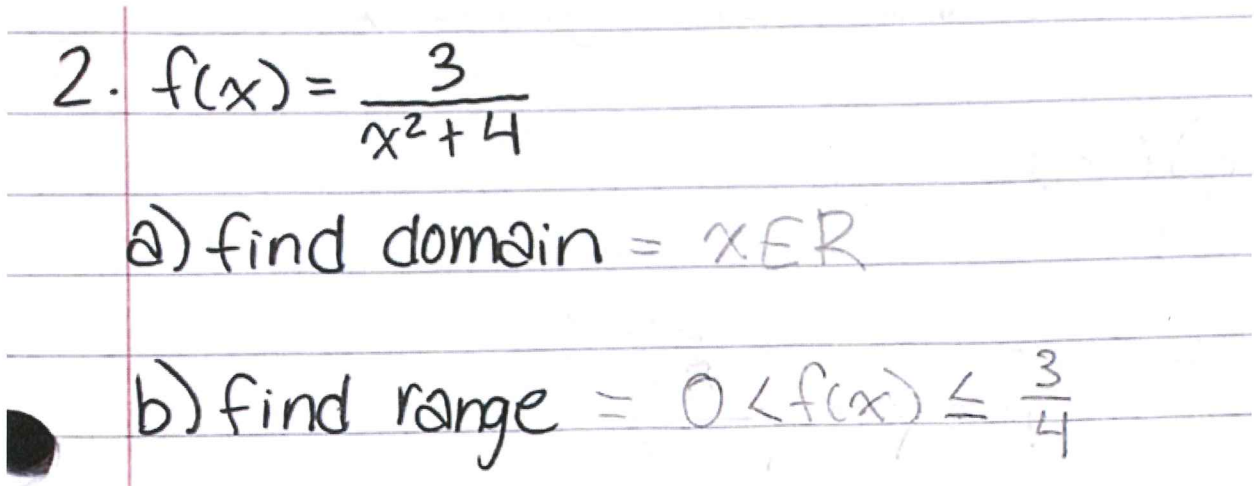
What do you notice about the ratio of words to symbols?

Who do you think would score a higher mark?

2. $f(x) = \frac{3}{x^2 + 4}$

a) find domain = $x \in \mathbb{R}$

b) find range = $0 < f(x) \leq \frac{3}{4}$



Student Solution #1

2. a) $f(x) = \frac{3}{x^2 + 4}$ finding domain of $f(x) \rightarrow x^2 \geq 0$

Step 1: equate denominator to zero so we can solve for x

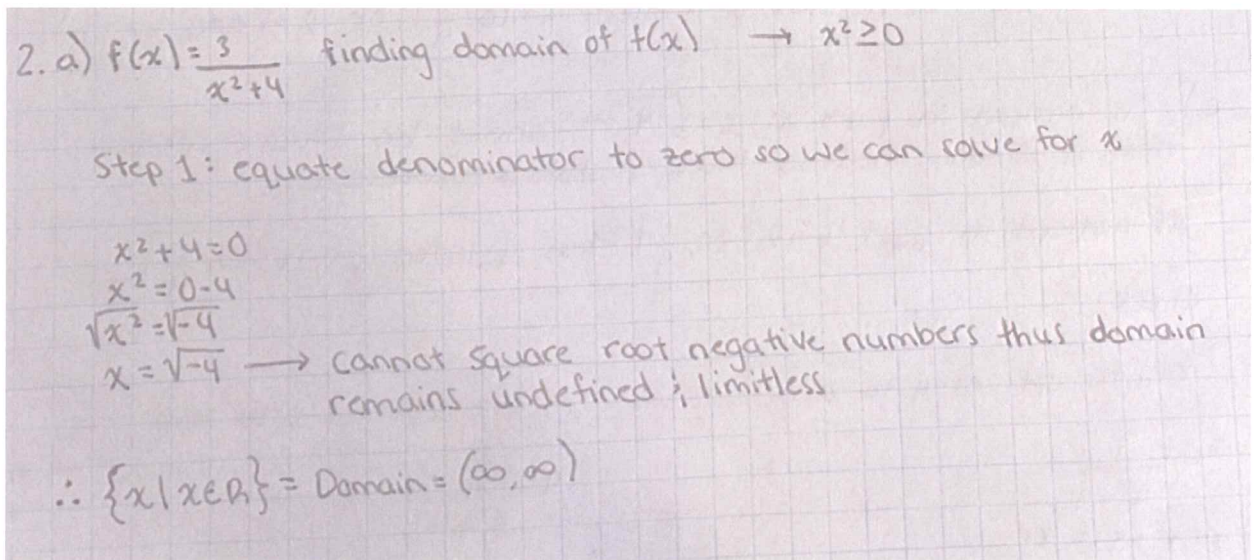
$$x^2 + 4 = 0$$

$$x^2 = 0 - 4$$

$$\sqrt{x^2} = \sqrt{-4}$$

$$x = \sqrt{-4} \rightarrow \text{cannot square root negative numbers thus domain remains undefined ; limitless}$$

$\therefore \{x | x \in \mathbb{R}\} = \text{Domain} = (-\infty, \infty)$

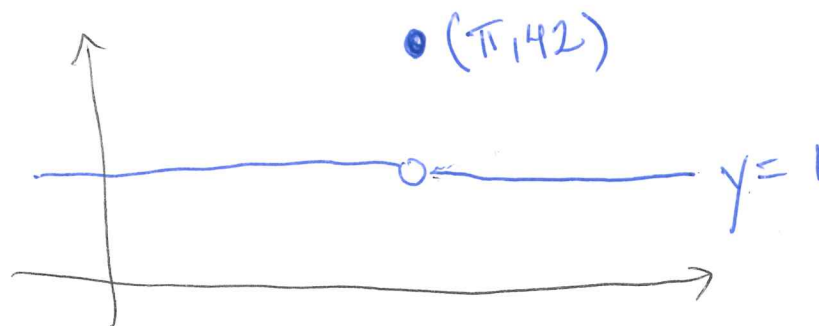


Student Solution #2

Question: A Piecewise Range

Write the domain and range of the function.

$$f(x) = \begin{cases} 1 & x \neq \pi \\ 42 & x = \pi \end{cases}$$



Domain: The ~~pieces~~ pieces $x \neq \pi$ and $x = \pi$ cover all possible real numbers. Therefore, the domain is $\mathbb{R}$.

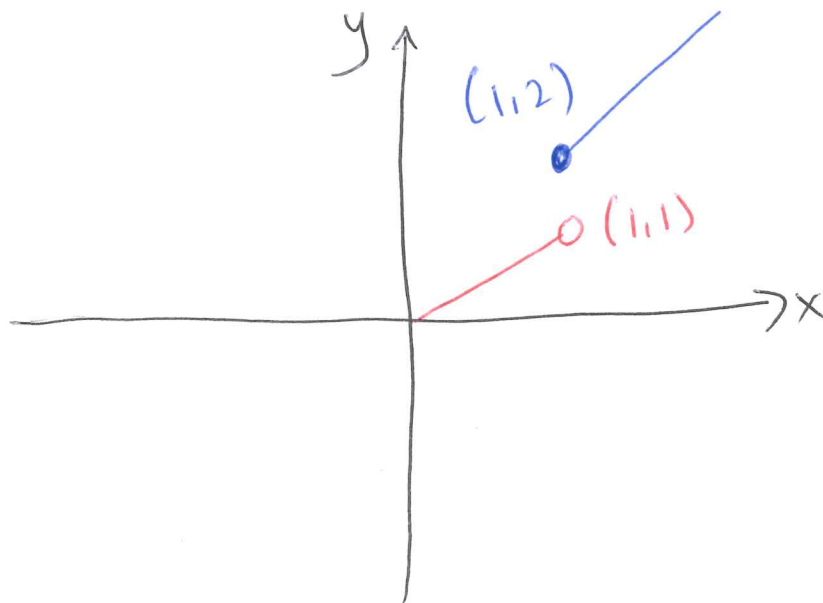
Range: The function only outputs $y=1$ and $y=42$. Therefore, the range is $\{1, 42\}$.

The curly brackets $\{ \dots \}$ denote sets. The set $\{1, 42\}$ has exactly two elements $y=1$ and $y=42$.

Activity: Solo Work: 2 min

Sketch the function:

$$f(x) = \begin{cases} x & x < 1 \\ x+1 & x \geq 1 \end{cases}$$

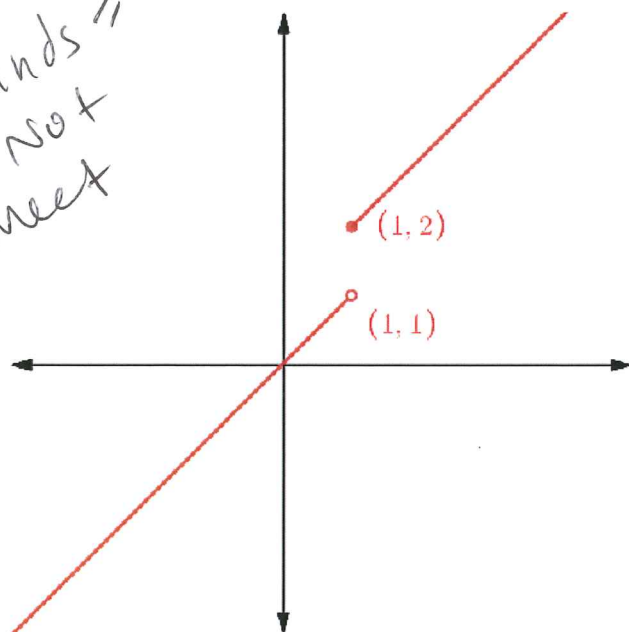
What do you notice about the values of $f(x)$ near $x = 1$?

Notice: The lines do not meet!
There is a jump/gap.

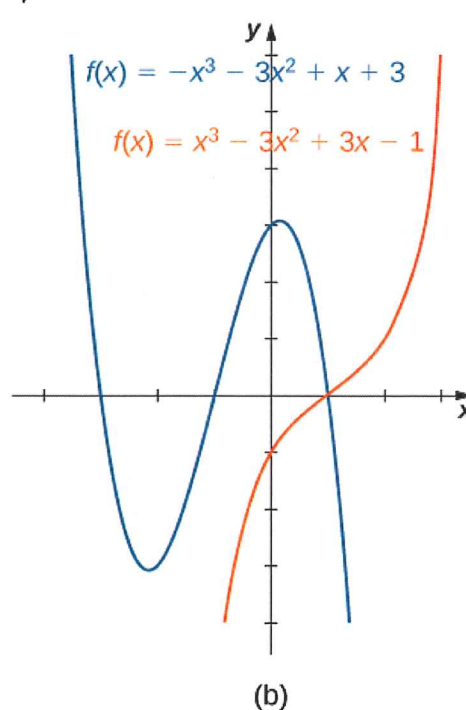
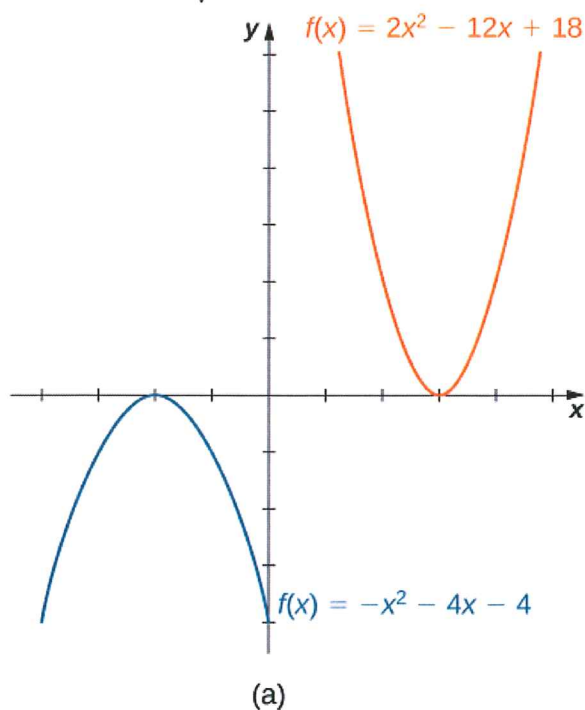
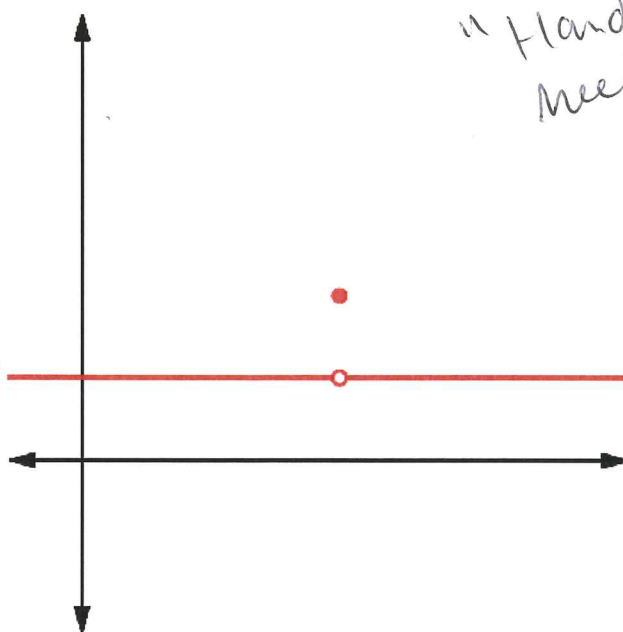
Remark: Three Similar Problems

One of the “super powers” of mathematics is its ability to explain multiple phenomena using the same language. This unifying process is called abstraction or generalization.

“Hands”
Do Not
Meet



“Hands”
Meet



End
Behavior

Definition: Limits

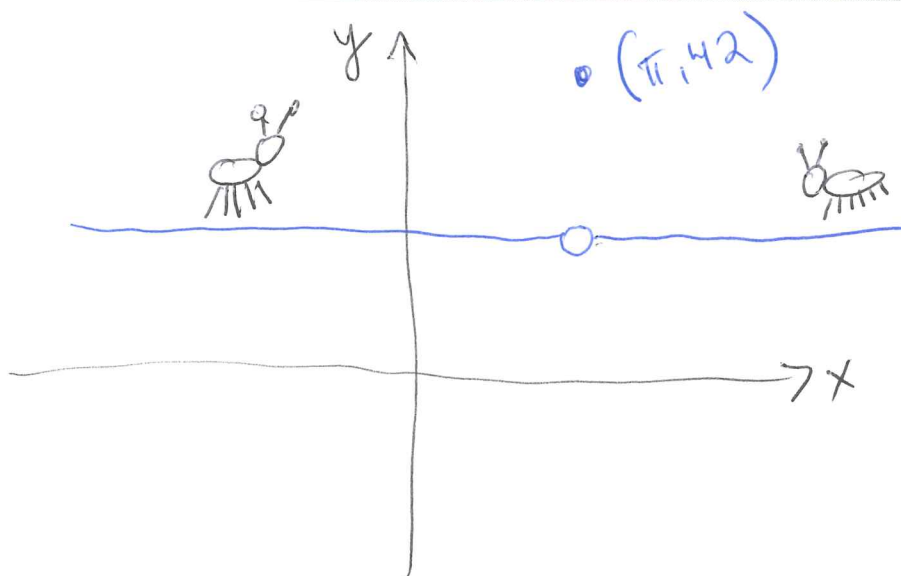
OS §1.2 Defn 2.3

If $f(x)$ gets closer to L as x gets closer to a then $\lim_{x \rightarrow a} f(x) = L$.
 We say that "the limit of $f(x)$ as x approaches a is L ".

Question: A Strange Limit

What is the limit of $f(x)$ as x approaches π ?

$$f(x) = \begin{cases} 1 & x \neq \pi \\ 42 & x = \pi \end{cases}$$



As we approach $x = \pi$, the graph is always at height $y = 1$.

We get: $\lim_{x \rightarrow \pi} f(x) = 1$

(The point $(\pi, 42)$ does not effect limit.)

Question: Limits Visually

Compute the limit $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$ by inspecting the graph on Desmos.

① We can't ^{always} find limits by plugging in.

$$\frac{\sin(0)}{0} = \frac{0}{0} \leftarrow \text{undefined.}$$

Inspecting the graph on Desmos, we see $\frac{\sin(x)}{x}$ is undefined at $x=0$.

When we zoom in near $x=0$, we see the graph looks like $y=1$.

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1.$$

What happens if $f(x)$ is defined at $x=0$?

We may have $\lim_{x \rightarrow 0} f(x) = f(0) \leftarrow \begin{cases} f \text{ is continuous} \\ \text{at } x=0 \end{cases}$

but sometimes $\lim_{x \rightarrow 0} f(x) \neq f(0)$

Question: Limits Algebraically

Compute the limit $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

First note: we cannot plug in $x=1$ because
$$\frac{1^2 - 1}{1 - 1} = \frac{0}{0} \leftarrow \text{undefined.}$$

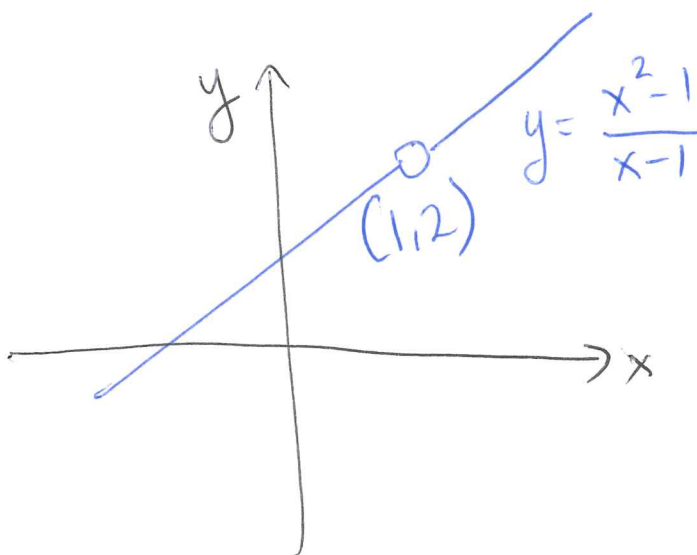
We can factor:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)}$$

$$= \lim_{x \rightarrow 1} x + 1$$

$$= 1 + 1 = 2$$

For all $x \neq 1$
we can cancel.





Definition: Two Sided Limits

Every number t has two sides:

the **left side** of t which we write t^- and the **right side** of t which we write t^+ .

The left hand limit is $\lim_{x \rightarrow t^-} f(x) = L$ if $f(x)$ approaches L when $x < t$.

The right hand limit is $\lim_{x \rightarrow t^+} f(x) = L$ if $f(x)$ approaches L when $x > t$.

Question: A Jump

The function $f(x)$ has a "jump" from $(1, 1)$ to $(1, 2)$.

$$f(x) = \begin{cases} x & x < 1 \\ x + 1 & x \geq 1 \end{cases}$$

Describe the nature of this jump using limits.

We have:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x \quad \# \quad x < 1 \text{ so we use } f(x) = x$$

$$= 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x + 1 \quad \# \quad x > 1 \text{ so we use } f(x) = x + 1$$

$$= 1 + 1 = 2$$

There is a jump from $(1, 1)$ to $(1, 2)$ because the left hand and right hand limits give those values.