Week 3: Limits and Derivatives

Definition: OS §1.2 Defn 2.3

If $f(x)$ gets closer to L as x gets closer to a then $\lim_{x \to a} f(x) = L$.
We say that "the limit of $f(x)$ as x approaches a is L".

Activity: A Limit Numerically (3 min)

Set your calculator to radian mode.

Complete the following table of values of $f(x) = \frac{\sin(x)}{x}$ for $x = 0.1, 0.01, 0.001$.

x	0.1	0.01	0.001
$f(x)$	~~[illegible]~~	0.99998	~~[illegible]~~

$0.\underbrace{99}_{\text{two nines}}8$ $\qquad$ $0.\underbrace{9999}_{\text{four nines}}8$ $\qquad$ $0.\underbrace{999999}_{\text{six nines}}8 \to 1$

There are double the number of nines as there are zeroes.
The limit goes "quickly"!

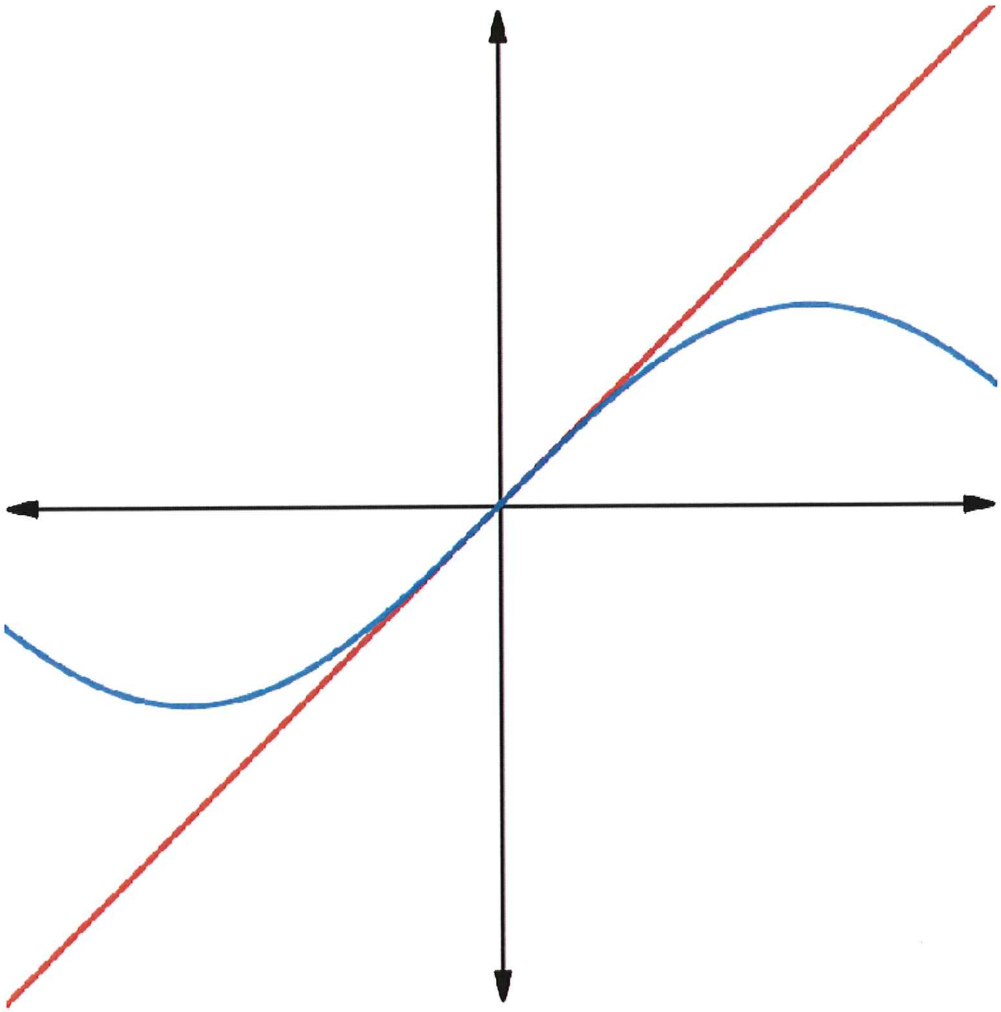
As $x \to 0$, we get $\frac{\sin(x)}{x} \to 1$

Question: Investigate A Limit Graphically

Graph $y = \sin(x)$ and $y = x$ on the same pair of axes and zoom in on the point $(0, 0)$. Write a paragraph describing what happens and how it related to $f(x) = \frac{\sin(x)}{x}$.

When we zoom in very close, $y=\sin(x)$ becomes a straight line. It becomes the line $y=x$. As x gets closer to zero, we see: $\frac{\sin(x)}{x}$ gets closer to one.

This also means: $\frac{x}{\sin(x)}$ gets closer to one.



https://www.desmos.com/calculator/dfbwtcocof

Theorem: OpenStax Theorem 2.2

Suppose that L is a real number.

$$\lim_{x \to a} f(x) = L \iff \left(\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L \right)$$

Comments:

- If both limits "go to the same number" then the limit is well-defined. / exists.
- Both sides of the limit need to match.

Example: A Life Sciences Example

Hemoglobin (Hb) is the iron-rich protein in blood cells that binds with oxygen (O2) in the lungs and exchanges it for carbon dioxide (CO2) in the tissues. In 1910, British physiologist Archibald Hill developed an empirical model to describe the binding of O_2 to Hb as[a]:

$$h(P) = \frac{P^k}{30^k + P^k} \qquad (k \geq 1)$$

where h is the proportion of hemoglobin molecules that are bound to O_2, P is the concentrations of O_2 measured as the partial pressure ($0 \leqslant P_{O_2} < \infty$), and k is the Hill coefficient.

1. Determine $\lim_{P \to 0} h(P)$ and interpret your results. We guess: $\lim_{P \to 0} h(P) = 0$.
2. Does the limit in part (1) change for different values of k? Explain. Probably not.
3. The half-saturation value $P_{50\%}$ is the concentration of oxygen at which the proportion of bound hemoglobin molecules reaches half its saturation value. Determine $P_{50\%}$.

[a]Hill AV. The possible effects of the aggregation of the molecules of haemoglobin on its dissociation curves. *J Physiol.* 1910; 40: iv–vii.

(1) $$\lim_{P \to 0} h(P) = \lim_{P \to 0} \frac{P^K}{30^K + P^K} = \frac{\left(\lim_{P \to 0} P^K\right)}{\left(\lim_{P \to 0} 30^K + P^K\right)} = \frac{0}{30^K + 0} = 0$$

(2) The value of [illegible] $\lim_{P \to 0} h(P)$ does not depend on K.

If $K < 0$ then we need to be careful.

$$\lim_{P \to 0} P^{-7} = \lim_{P \to 0} \frac{1}{P^7}$$ ← This can blow-up or not exist.

(3) The largest possible value of $h(P)$ is when every hemoglobin is fully saturated with oxygen and $h(P) = 1$.

We now calculate $P_{50\%}$ such that $h(P_{50\%}) = \frac{1}{2} = \frac{1}{2} \cdot 1$

$$\frac{1}{2} = \frac{P^k}{30^k + P^k} \Leftrightarrow \frac{1}{2}(30^k + P^k) = P^k$$

$$\Leftrightarrow \frac{1}{2} \cdot 30^k = \left(1 - \frac{1}{2}\right) P^k$$

$$\Leftrightarrow \frac{1}{2} 30^k = \frac{1}{2} P^k$$

$$\Leftrightarrow 30^k = P^k$$

$$\Leftrightarrow 30 = P.$$

Therefore $P_{50\%} = 30$. In this model, a person under 30 partial pressures of O_2 will have half their Hb saturated with oxygen.

Question: An Infinite Limit

Use a table of values to support the following statement:

$$\text{“}\lim_{x\to 0} \frac{1}{x} \text{ does not exist.”}$$

Activity: Solo Work: (2 min)

Use a table of values to support the following statement:

$$\text{“}\lim_{x\to 0} \frac{1}{x^2} \text{ is infinite.”}$$

Consider the following table of values.

x	$-1/10$	$-1/100$	$-1/1000$	$1/1000$	$1/100$	$1/10$
$1/x$	-10	-100	-1000	1000	100	$\frac{1}{1/10} = 10$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

We have $\lim_{x\to 0^-} \frac{1}{x} \neq \lim_{x\to 0^+} \frac{1}{x}$ and so $\lim_{x\to 0} \frac{1}{x}$ does not exist.

x	$-1/10$	$-1/100$	$-1/1000$	$1/1000$	$1/100$	$1/10$
$1/x^2$	100	10 000	1000 000	1000 000	10000	100

∞

This suggests $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Remark: Existence and Infinity: A Quote from Our Book

It is important to understand that when we write statements such as $\lim_{x\to a} f(x) = +\infty$ or $\lim_{x\to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x\to a} f(x) = +\infty$, we always write $\lim_{x\to a} f(x) = +\infty$ rather than $\lim_{x\to a} f(x)$ DNE. **[OpenStax §2.2 p.146]**

Activity: Discuss with Your Neighbour (3 min)

- What do you make of this quote?
- Do infinite limits exist? Why or why not?

Infinite limits do not exist,
BUT we write $\lim_{x\to a} f(x) = \boxed{+\infty}$
if: $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \infty$.

And similar stuff for $\lim_{x\to a} f(x) = -\infty$

Infinity is NOT a number,
it is a process.

Question: Compute a Limit: Plugging-In Doesn't Help

Evaluate the limit $\lim_{x\to 1} \dfrac{x^2-3x+2}{x-1}$.

If we plug in to the limit, we get:

$$\frac{1^2-3\cdot 1+2}{1-1} = \text{“}\frac{0}{0}\text{”} \leftarrow \text{Indeterminate or undefined}$$

We need to do $\boxed{\text{algebra}}$:

$$\lim_{x\to 1} \frac{x^2-3x+2}{x-1} = \lim_{x\to 1} \frac{(x-1)(x-2)}{x-1}$$

$$= \lim_{x\to 1} x-2 = 1-2 = -1.$$

For ease of reference, here are the limit laws.

Theorem: The Limit Laws

1. $\lim_{x\to a} c = c$
2. If n is a non-negative integer, then $\lim_{x\to a} x^n = a^n$.
3. $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
4. $\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
5. $\lim_{x\to a} f(x) \times g(x) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
6. If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$.
7. If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ provided both limits exist.

Note: You will be provided these laws on the term test.

Question: Compute a Limit Using Known Data

Use the limit laws to evaluate

$$\lim_{x\to 1}(x^2 - x + 1)f(x)$$

if you know $\lim_{x\to 1} f(x) = 3$.

Revisit?

Definition: Slope (OpenStax §1.2 Eq 1.3)

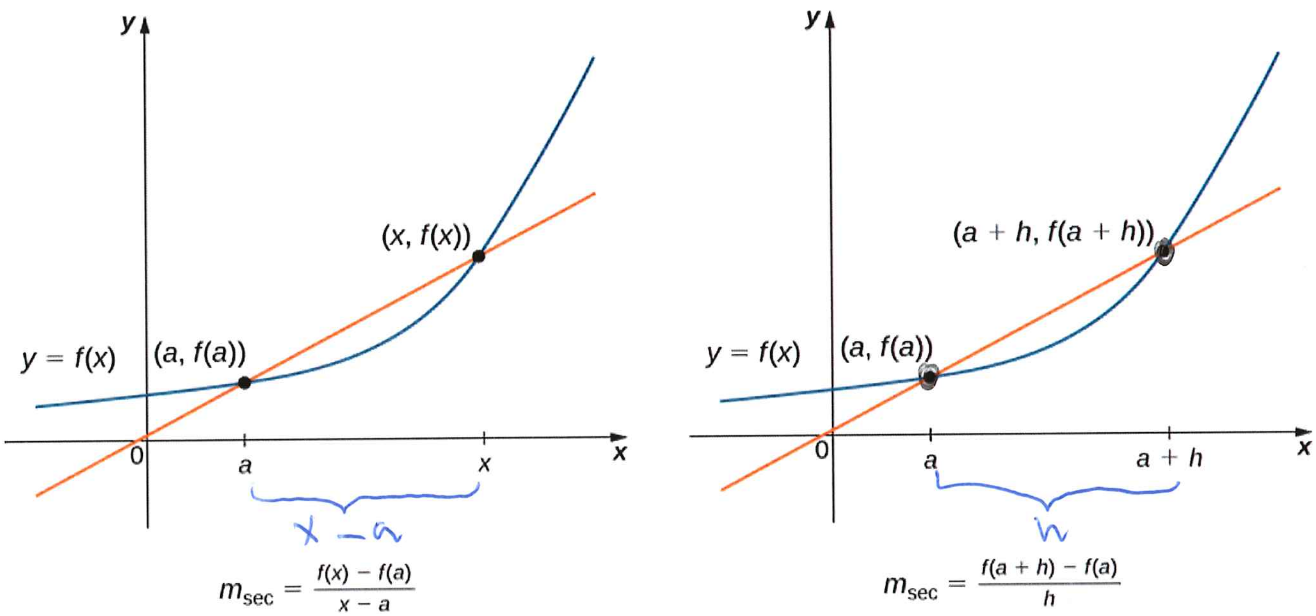
The **slope** of a line L through points (x_0, y_0) and (x_1, y_1) is:

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = m$$

Definition: Secant Slope

The **secant line** between two points $(a, f(a))$ and $(x, f(x))$ is a line segment joining those points. The **secant slope** is:

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$



OpenStax §3.1 Fig 3.3

Let the points be $(a, f(a))$ and $(a+h, f(a+h))$

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - (a)} = \frac{f(a+h) - f(a)}{h}$$

Definition: Derivatives: OpenStax Pg. 220

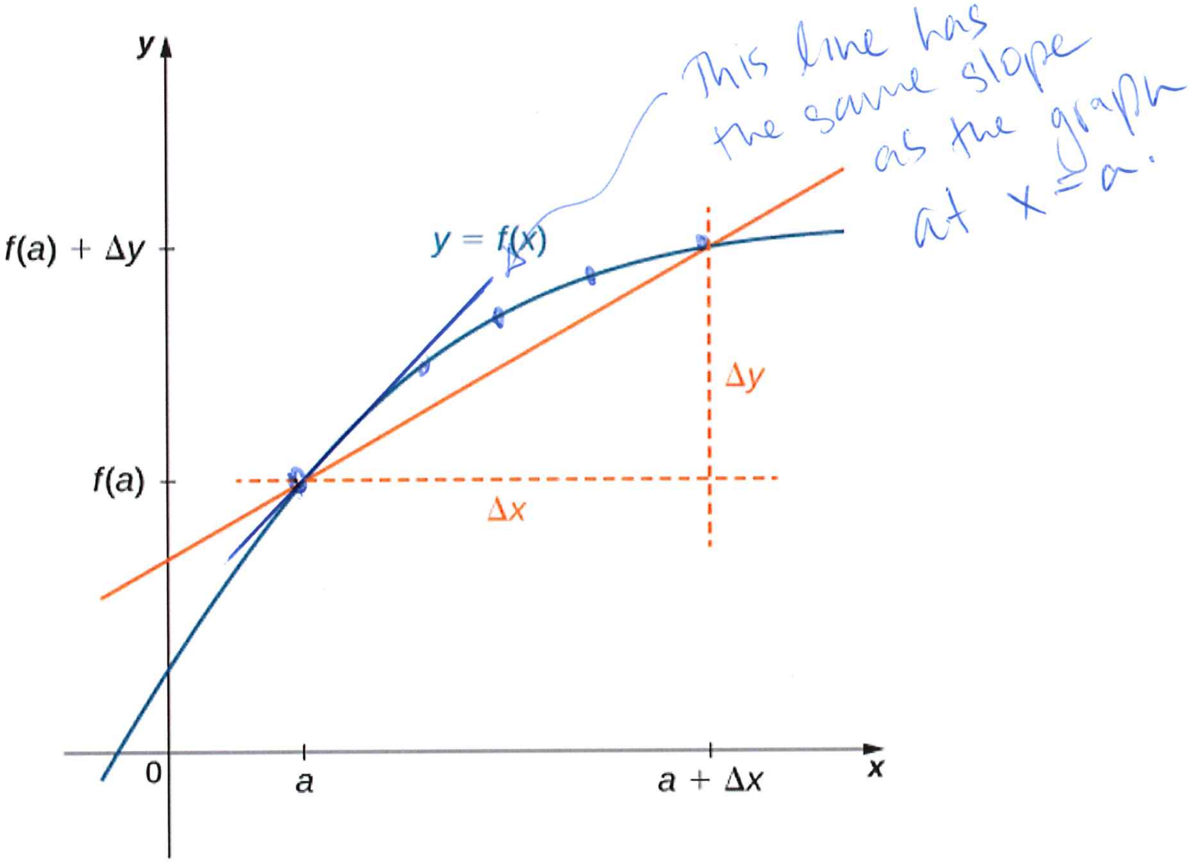
We define the **derivative** of $f(x)$ at $x = a$ to be:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \text{“ } f \text{ prime at } a \text{ ”}$$

We will also use the following **Leibniz notation**:

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{“ dee } y \text{ dee } x \text{ ”}$$

The slope of $y = f(x)$ at $x = a$.



$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

OpenStax Pg. 234

Remark: Do we have to learn both notations?

Yes, absolutely and without a doubt. You need to be comfortable writing and interpreting both notations.

“I will learn Leibniz notation.”

Question: A Derivative from the Definition

Find the slope of $f(x) = x^2 + x$ at $a = 1$ (using the definition of the derivative.)

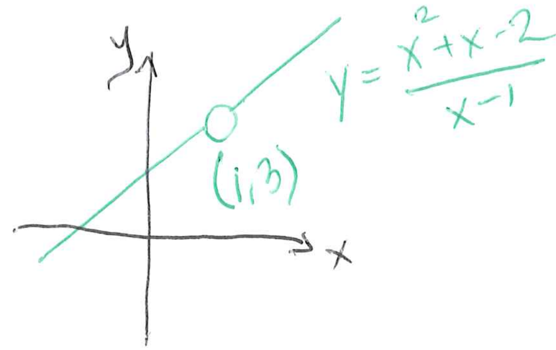
We calculate

$$f'(a) = f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{x \to 1} \frac{(x^2 + x) - (1^2 + 1)}{x - 1} = \lim_{x \to 1} \boxed{\frac{x^2 + x - 2}{x - 1}} \quad \frac{1+1-2}{1-1} = \frac{``0"}{0}$$

$$= \lim_{x \to 1} \frac{(x+2)(x-1)}{(x-1)} = \lim_{x \to 1} (x+2) \quad \boxed{\text{for } x \neq 1}$$

$$= 3$$



Therefore, $f'(1) = 3$.

Remark: Frequently Asked Questions

Do we always have to do it this way?
No – We're doing things this way for now, so that I can teach you what a derivative really means. This is a really awkward way to compute a derivative.

Must we always show this much work?
Sort of – You should always show your work, and explain each step.

Can we use derivative rules?
Yes – Once we learn them.

$f(x) = x^2 + x$

$\Rightarrow f'(x) = 2x + 1$

Example: Working Backwards to a Derivative

Consider the following limit:

$$\lim_{h\to 0} \frac{e^h - 1}{h}$$

Which function $f(x)$ is this the derivative of? At what point a? What is the value of $f'(a)$?
Silly Joke: This limit is really Canadian, e^h?

We want this to match the format

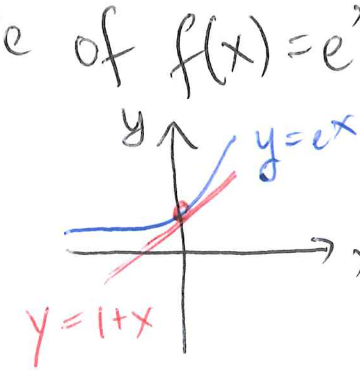
$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

We write the limit in a way that makes $f(x)$ and a more clear.

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{e^h - e^0}{h} \quad \# \; e^0 = 1$$

$$= \lim_{h\to 0} \frac{e^{0+h} - e^0}{h} \quad \# \; 0+h=h$$

This given limit is the derivative of $f(x)=e^x$ at $a=0$. For now, we would have to examine the graph very carefully to get $f'(0)=1$.



Activity: Micro-Assignment (5 min)

Find the slope of $f(x) = \frac{1}{x}$ using the definition of the derivative where $a \neq 0$.

use the def^n $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

We need $a \neq 0$ so that we can divide here.

$$= \lim_{h \to 0} \left(\frac{\frac{1}{a+h} - \frac{1}{a}}{h} \right) \left(\frac{a(a+h)}{a(a+h)} \right)$$

$$= \lim_{h \to 0} \frac{a - (a+h)}{h a (a+h)} = \lim_{h \to 0} \frac{-h}{h a (a+h)}$$

$\boxed{\text{for } h \neq 0}$

$$= \lim_{h \to 0} -\frac{1}{a(a+h)}$$

+2 min

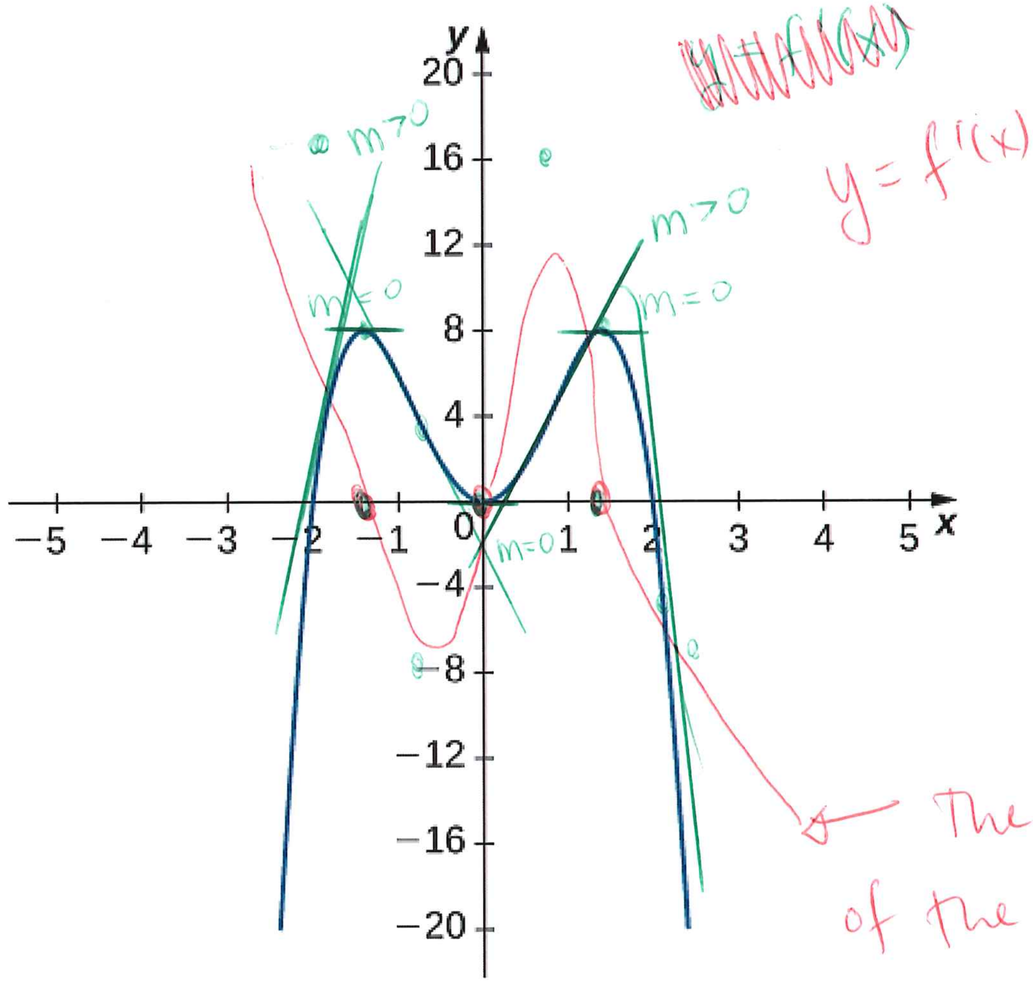
$$= -\frac{1}{a(a+0)} = -\frac{1}{a^2}$$

Definition: The Derivative Function

The **derivative** of $f(x)$ is the function $f'(x)$. It measures the slope of $y = f(x)$ at each point $(x, f(x))$.

Example: Sketching a Slope Visually

Use the following graph of $y = f(x)$ to sketch the derivative function $y = f'(x)$.



The slope of the original graph.

OpenStax §3.2 Q65.