Functions and Slope

Definition (OpenStax Pg. 220)

We define the derivative $f(x)$ at $x = a$ to be:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \text{“ } f \text{ prime at } a \text{ ”}$$

Definition (OpenStax Pg. 234)

Leibniz: (*) $$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{“ dee } y \text{ dee } x \text{ ”}$$ SLOPE

We will use $\frac{d}{dx}[\cdot]$ as a way of writing “take the derivative”.

$$\frac{d}{dx}[\text{stuff}] = \text{derivative of stuff}$$

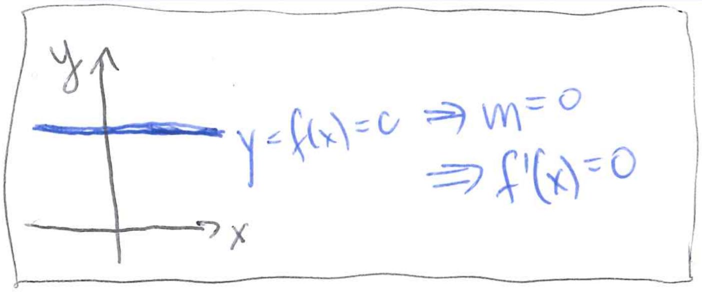
Notes

Constants!

Theorem (OpenStax §3.3 Theorem 3.2)

$$\frac{d}{dx}[c] = 0$$

Alternatively, if $f(x) = c$ is constant and does not change, then $f'(x) = 0$.



The algebra/limits version.

We calculate:

$\boxed{\text{Let } f(x) = c \text{ for all } x.}$

This gives:

$$\frac{d}{dx}[c] = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h}$$

$$= \lim_{h\to 0} 0 \quad (\text{for } h \neq 0)$$

$$= 0$$

Notes

Powers

Theorem (OpenStax §3.3 Theorem 3.3)

If n is a number,

$$\boxed{\frac{d}{dx}\left[x^n\right] = nx^{n-1}}$$

(We consider just $n=3$.) Let $f(x)=x^3$. We get:

$$\frac{d}{dx}\left[x^3\right] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\left(\cancel{x^3} + 3x^2h + 3xh^2 + h^3\right) - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{h\left(3x^2+3xh+h^2\right)}{h} = \text{"}\frac{0(\ldots)}{0}\text{"}$$

$$= \lim_{h\to 0} 3x^2+3xh+h^2 \ (\text{for } h\neq 0) = 3x^2 = \boxed{3x^{3-1}}$$

Notes

Aside: We expand

$$(x+h)^3 = (x+h)(x+h)(x+h)$$

$$= (x+h)\left(x^2 + xh + hx + h^2\right)$$

$$= (x+h)\left(x^2 + 2xh + h^2\right)$$

$$= x^3 + 2x^2h + xh^2 + hx^2 + 2xh^2 + h^3$$

$$= x^3 + 3x^2h + 3xh^2 + h^3$$

$$\boxed{\text{"}nx^{n-1}h\text{"}}$$

Why do we write

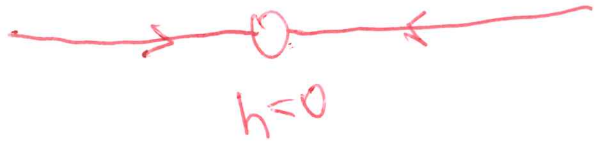
$$\lim_{h \to 0} \frac{h(\ldots)}{h}$$

$$= \lim_{h \to 0} (\ldots) \text{ for } h \neq 0$$

and then let $h=0$?!

The big deal here is that cancelling h from top and bottom changes the function. The function changes because it becomes defined at $h=0$. (The domain changes to include $h=0$.)

The amazing fact about limits is they let us evaluate functions at points outside their domain.



Linearity

DISCUSS 2min.

Theorem (OpenStax §3.3 Theorem 3.4)

For any numbers a and b: $\boxed{\frac{d}{dx}[af(x)+bg(x)] = af'(x)+bg'(x).}$

Sum Rule ($a=b=1$)

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

Difference Rule ($a=1$ *and* $b=-1$)

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

Constant Multiple Rule ($\square$)

$$\frac{d}{dx}[kf(x)] = kf'(x)$$

?

Take $a=k$.
$b=0$.

We calculate:

$$\frac{d}{dx}\left[af(x)+bg(x)\right] = \lim_{h\to 0}\frac{\left[af(x+h)+bg(x+h)\right]-\left[af(x)+bg(x)\right]}{h}$$

Notes

$$= \lim_{h\to 0}\left[\frac{a(f(x+h)-f(x))}{h} + \frac{b(g(x+h)-g(x))}{h}\right]$$

$$= a\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + b\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= af'(x)+bg'(x)$$

Add more time.

13:45

Polynomials!

Activity (Micro-Assignment (5 min))

Compute the derivative of $f(x) = x^3 - 2x + 1$.
Either: use the limit definition, or carefully explain every step using the rules from this class.

Limit Def^n

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[(x+h)^3 - 2(x+h) - 1\right] - \left[x^3 - 2x + 1\right]}{h}$$

Deriv Rules

$$\frac{d}{dx}\left[x^3 - 2x + 1\right]$$

$$= \frac{d}{dx}[x^3] - \frac{d}{dx}[2x] + \frac{d}{dx}[1] \quad \text{\# sum and diff.}$$

$$= 3x^2 - \frac{d}{dx}[2x] + 0 \quad \text{\# const and polynomials}$$

$$= 3x^2 - 2\frac{d}{dx}[x] \quad \text{\# const. multiple}$$

$$= 3x^2 - 2 \cdot 1 = 3x^2 - 2.$$

Notes

$= \ldots$ (Algebra)

$= \lim_{h \to 0} (\ldots)$ for $h \neq 0$.

$= 3x^2 - 2.$

$$\frac{d}{dx}[x^3] = 3x^{3-1} = 3x^2$$

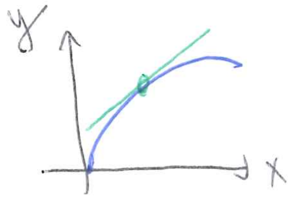
Using Derivatives

Definition (OpenStax Pg. 217)

The tangent slope to $y = f(x)$ is:

$$m_{tan} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The tangent line to $y = f(x)$ at $x = a$ has slope m_{tan} and passes through $(a, f(a))$.



The tangent line has exactly the same slope as $y = f(x)$ at $x = a$, and passes through $(a, f(a))$.

Notes

Using Derivatives

Question

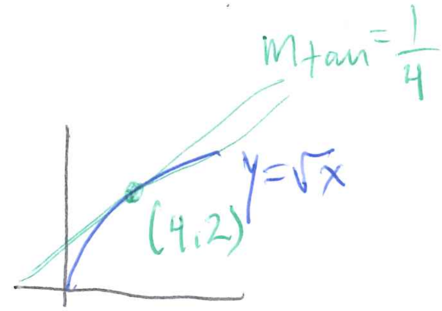
Find the tangent slope of $f(x) = \sqrt{x}$ at $a = 4$.

$$m_{tan} = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{x - 4} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{(\sqrt{x} + \sqrt{4})(\sqrt{x} - \sqrt{4})}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x} + \sqrt{4}} \quad (\text{for } x \neq 4)$$

$$= \frac{1}{\sqrt{4} + \sqrt{4}} = \frac{1}{2+2} = \frac{1}{4}$$



Notes

Using Derivatives

Question

Find the tangent line of $f(x) = \sqrt{x}$ *at* $a = 4$ *assuming* $m_{tan} = \frac{1}{4}$.

The point $(4, \sqrt{4}) = (4,2)$ is on ~~the~~ both $y = \sqrt{x}$ and the tangent line. We use point-slope format and get:

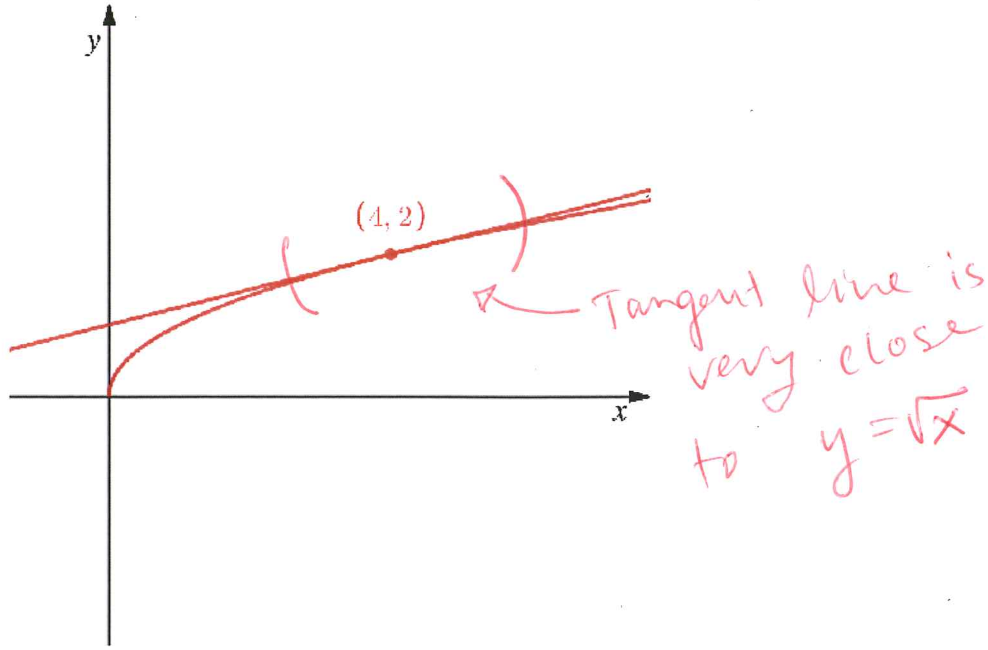
$$y - 2 = m_{tan}(x-4)$$

$$y - 2 = \frac{1}{4}(x-4) = \frac{1}{4}x - 1$$

$$\iff y = \frac{1}{4}x + 1$$

Notes

Using Derivatives



https://www.desmos.com/calculator/luadl4c0iy

Notes

Week 8 Sneak Peek!

Replace hard $\sqrt{x}$ by $\frac{1}{4}x+1$ easier.

Question

Use the tangent line $y = \frac{1}{4}x + 1$ of $y = \sqrt{x}$ at $a = 4$ to approximate $\sqrt{4.242}$.

What's the idea here? $4.242 \approx 4.000$.
Near the point of tangency, $\frac{1}{4}x + 1 \approx \sqrt{x}$.

$$\frac{1}{4}(4.242) + 1 = \underline{2.06}05 \qquad \sqrt{4.242} = \underline{2.059}6\ldots$$

That's a pretty good approximation!

Notes

Using Derivatives

Question

Where is the tangent line of $y = x^3 - 3x$ horizontal?

- Horizontal tangent $\iff m_{tan} = 0$
- $m_{tan} = \frac{dy}{dx} = \frac{d}{dx}[x^3 - 3x] = 3x^2 - 3 = 3(x^2 - 1) = 0$

$\iff 3(x-1)(x+1) = 0$

$\iff x = 1$ or $x = -1$.

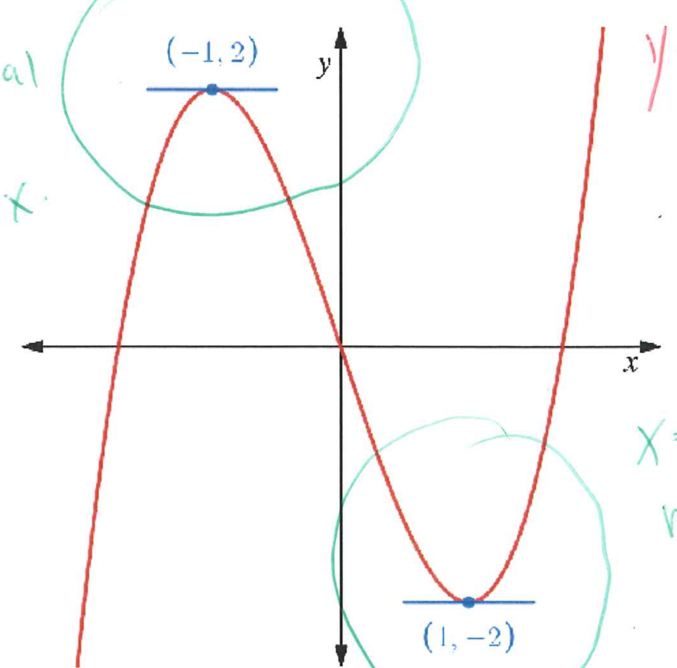
The tangent line at $x = -1$ or $x = 1$ is horizontal.

Notes

Using Derivatives

$y = x^3 - 3x$

$x = -1$ is a local maximum of $f(x) = x^3 - 3x$.



$x = 1$ is a local minimum of $f(x) = x^3 - 3x$

https://www.desmos.com/calculator/djnzupah2i

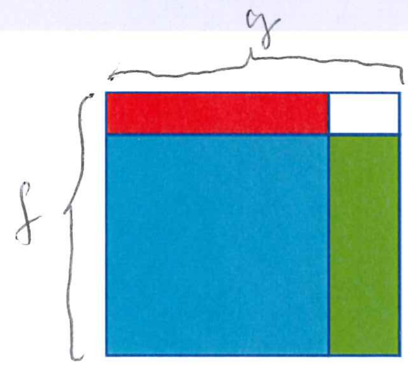
Local means "near the value."

Notes

Product Rule

Theorem (OpenStax §3.3 Pg. 253)

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$



$$\frac{d}{dx}[fg] = f'g + fg'$$

Add and subtract additional terms.

Notes

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{g(x+h)\,[f(x+h)-f(x)]}{h} + \frac{f(x)\,[g(x+h)-g(x)]}{h}\right]$$

$$= g(x+0)f'(x) + f(x)g'(x) = f'(x)g(x) + f(x)g'(x).$$

An Absolute Surprise

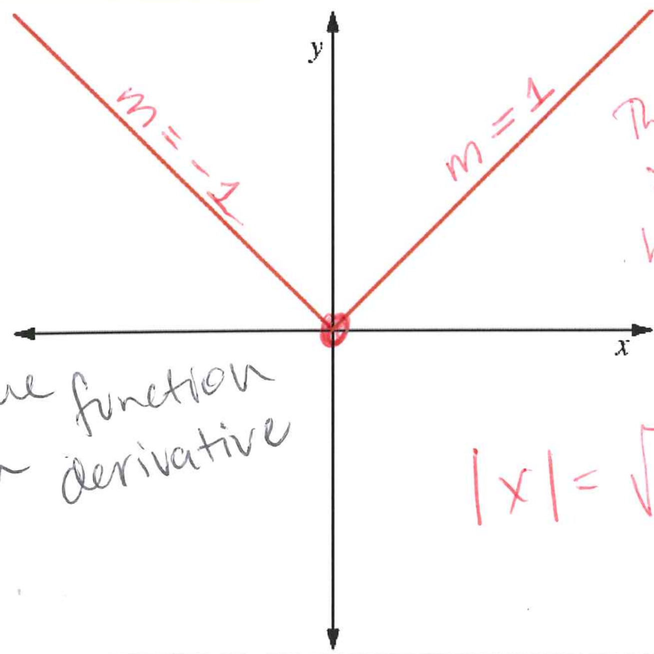
Definition

The <u>absolute value</u> function $f(x) = |x|$ is defined as:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$\pi = |-\pi| = -(-\pi)$

Why would anyone define this thing? It measures the distance from x to 0.



The distance from x to 0 must be positive: $|x|$.

The absolute value function does not have a derivative at $x = 0$.

$|x| = \sqrt{x^2}$

Notes

An Absolute Surprise

Question

What's the slope of $y = |x|$ at $(x, y) = (0, 0)$?

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

To evaluate this limit, we need to calculate $\lim_{h \to 0^+} \frac{|h|}{h}$ ad $\lim_{h \to 0^-} \frac{|h|}{h}$. We calculate $\lim_{h \to 0^+} \frac{|h|}{h}$ first.

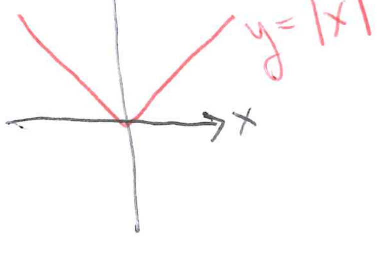
$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 \ (\text{for } h \neq 0) = 1.$$

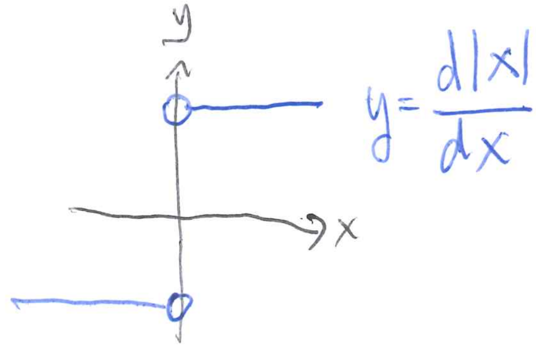
Notes

On the other side:

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 \ (\text{for } h \neq 0) = -1.$$

This gives $\lim_{h \to 0^+} \neq \lim_{h \to 0^-}$ ad so $\lim_{h \to 0}$ does not exist.





On the TT1 Description
"Limits as Derivatives"

$$\lim_{h\to 0} \frac{e^h - 1}{h}$$

What is This the Derivative Of?

Question

The $\lim_{h\to 0} \frac{\sqrt{h^2 - h}}{h}$ *represents which of the following derivative calculations?*

1. $f'(0)$ *where* $f(x) = \sqrt{x^2 - x}$
2. $g'(x)$ *where* $g(x) = \sqrt{x^2}$
3. $h'(1)$ *where* $h(x) = x^2 - x$
4. $u'(-1)$ *where* $u(x) = x^2$

(5 min)

$$f'(x) = \boxed{\lim_{h\to 0}} \boxed{\frac{\boxed{f(x+h)} - \boxed{f(x)}}{h}}$$

Notes

$$\lim_{h\to 0} \frac{\sqrt{h^2-h}}{h} = \lim_{h\to 0} \frac{\sqrt{h^2-h} - 0}{h} = \lim_{h\to 0} \frac{\sqrt{h^2-h} - \sqrt{0^2-0}}{h}$$

$$= \lim_{h\to 0} \frac{\boxed{\sqrt{(0+h)^2-(0+h)}} - \boxed{\sqrt{0^2-0}}}{h}$$

= "the derivative of $\sqrt{x^2 - x}$ at $x = 0$"

= 1.