Quotient Rule

Theorem (OpenStax §3.6 Pg. 255)

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\boxed{f'(x)g(x)} - \boxed{f(x)g'(x)}}{[g(x)]^2}$$ ← *The messiest derivative rule!*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}\left(\frac{g(x+h)g(x)}{g(x+h)g(x)}\right)$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)}{\cancel{g(x+h)}}\cancel{g(x+h)}g(x) - \frac{f(x)}{\cancel{g(x)}}g(x+h)\cancel{g(x)}}{h\,g(x+h)g(x)}$$

This cancellation is tricky.

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x+h)g(x)}$$

Notes

$$= \lim_{h\to 0} \frac{\boxed{f(x+h)g(x) - f(x)g(x)} + \boxed{f(x)g(x) - g(x+h)f(x)}}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0}\left[\frac{1}{g(x+h)g(x)}\,\frac{f(x+h)g(x) - f(x)g(x)}{h}\right] + \left[\frac{1}{g(x+h)g(x)}\,\frac{f(x)g(x) - g(x+h)f(x)}{h}\right]$$

Looks like f'

Looks like g'

$$= \lim_{h \to 0} \left[\frac{1}{g(x)g(x+h)} \frac{f(x+h)g(x) - f(x)g(x)}{h} \right.$$

$$\left. - \left[\frac{1}{g(x)g(x+h)} \frac{g(x+h)f(x) - g(x)f(x)}{h} \right] \right]$$

$$= \lim_{h \to 0} \frac{1}{g(x)g(x+h)} g(x) \left[\frac{f(x+h) - f(x)}{h} \right]$$

$$- \frac{1}{g(x)g(x+h)} f(x) \left[\frac{g(x+h) - g(x)}{h} \right]$$

$$= \frac{1}{[g(x)]^2} \left[g(x) f'(x) \right] - \frac{1}{[g(x)]^2} \left[f(x) g'(x) \right]$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Yay! The Quotient Rule.

$$\lim_{h \to 0} \frac{1}{g(x)g(x+h)} = \frac{1}{g(x)g(x+0)} = \frac{1}{[g(x)]^2}$$

Quotient Rule

Question

Apply the quotient rule to $k(x) = \dfrac{5x^2}{4x+3}$.

We calculate:

$$k'(x) = \frac{\frac{d}{dx}\left[5x^2\right](4x+3) - 5x^2\frac{d}{dx}\left[4x+3\right]}{(4x+3)^2}$$

$$= \frac{10x(4x+3) - 5x^2(4)}{(4x+3)^2}$$

Notes

$$= \frac{40x^2 + 30x - 20x^2}{(4x+3)^2}$$

$$= \frac{20x^2 + 30x}{(4x+3)^2} = \frac{10x(2x+3)}{(4x+3)^2}$$

$$= \frac{20x^2 + 30x}{16x^2 + 24x + 9}$$

Apply the Quotient Rule

(2 min)

Question

What is the derivative of $f(x) = \dfrac{x+4}{x+1}$ *at the point* $x = 2$?

1. $f'(x) = -3$
2. $f'(x) = -\frac{1}{3}$
3. $f'(x) = 2$
4. *f does not have a derivative at $x = 2$.*

$$f'(x) = \frac{\frac{d}{dx}[x+4](x+1) - (x+4)\frac{d}{dx}[x+1]}{(x+1)^2}$$

$$= \frac{1(x+1) - (x+4)1}{(x+1)^2} = \frac{-3}{(x+1)^2}$$

Notes

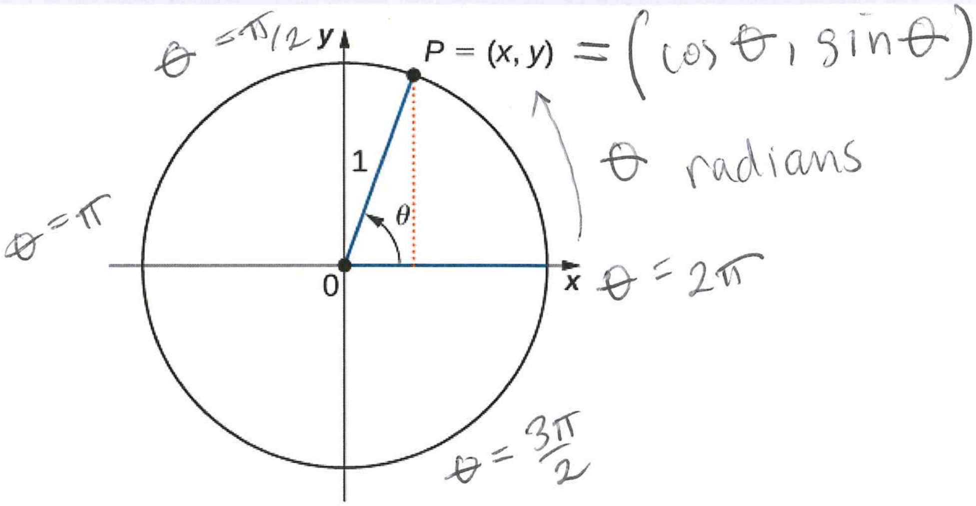
This gives:

$$f'(2) = \frac{-3}{(2+1)^2} = -\frac{3}{9} = -\frac{1}{3}$$

The Trigonometric Functions sin and cos

Question

How are $\sin(x)$ *and* $\cos(x)$ *related?*



OpenStax §1.3 Fig 1.31

$$(x, y) = (\cos(\theta), \sin(\theta)) \iff \boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$$

Notes

Various other Trig Functions

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

"cotangent"

$\sec\theta = \dfrac{1}{\cos\theta}$

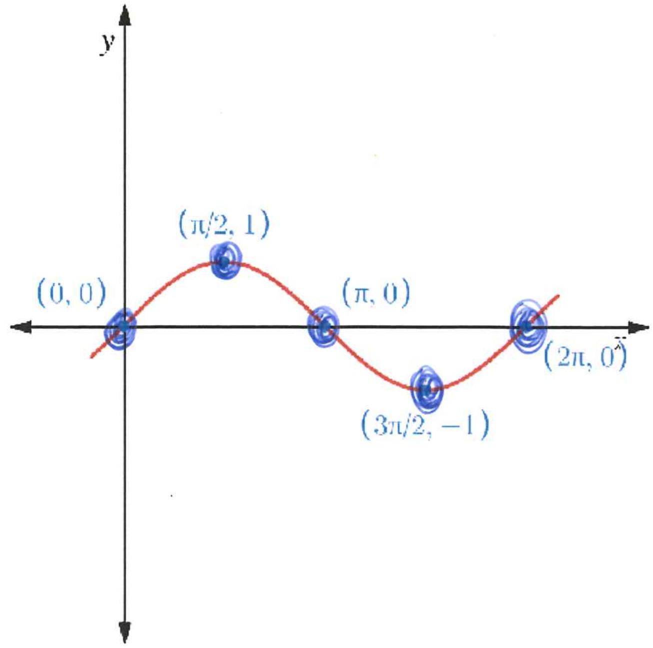
"secant"

$\csc\theta = \dfrac{1}{\sin\theta}$

"cosecant"

These are all "just" ways of re-writing $\sin\theta$ and $\cos\theta$.

Tangents to the sin Curve

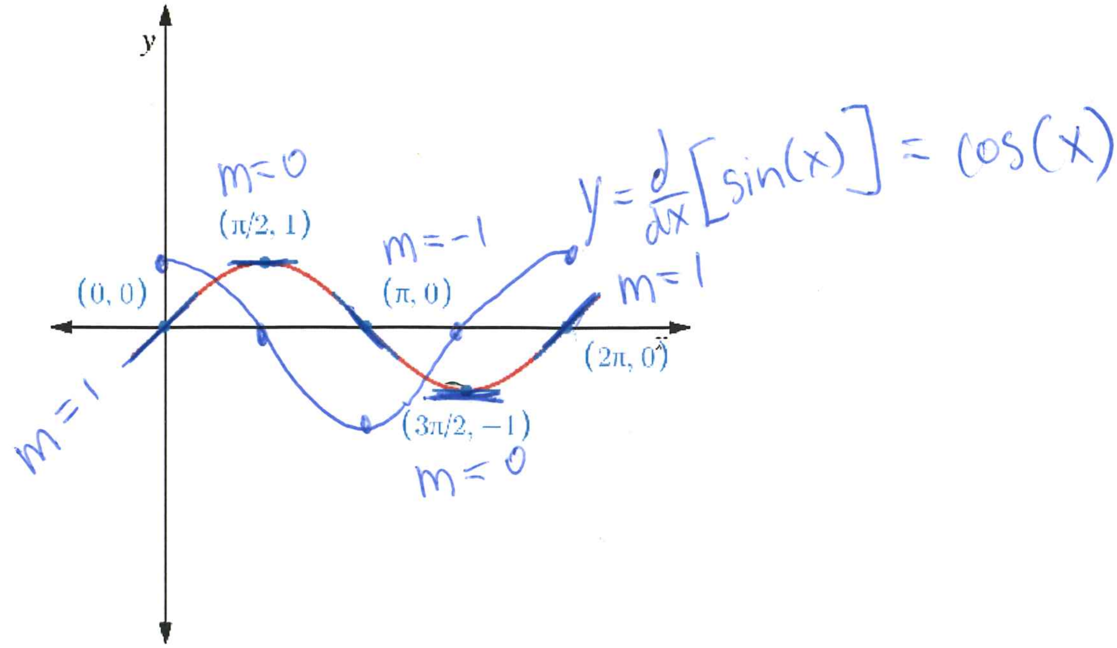


$\sin(0) = 0$
$\sin(\frac{\pi}{2}) = 1$
$\sin(\pi) = 0$
$\sin(\frac{3\pi}{2}) = -1$
$\sin(2\pi) = 0.$

$y = \sin(x)$ with important points highlighted

Notes

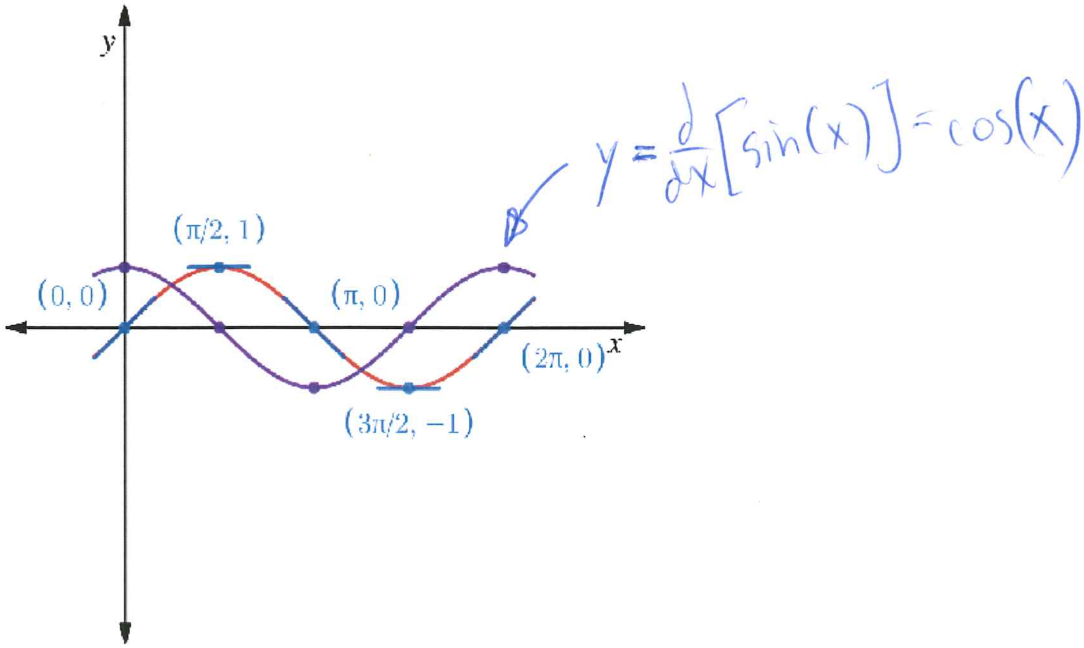
Tangents to the sin Curve



$y = \sin(x)$ with tangent lines at important points

Notes

Tangents to the sin Curve



$y = \sin(x)$ with derivative curve

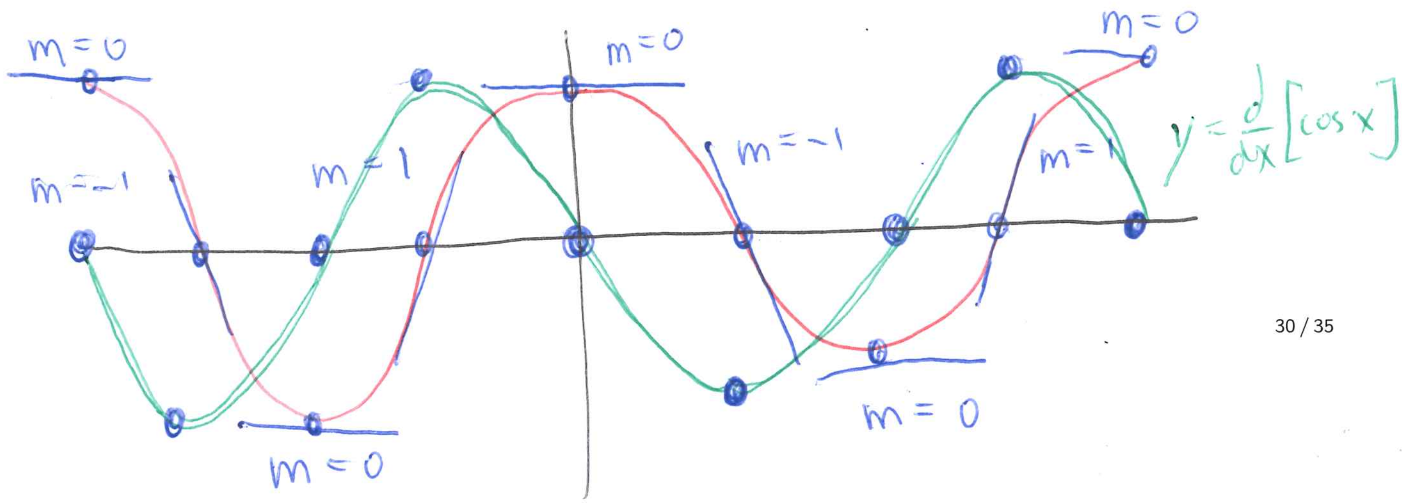
Notes

Tangents to the sin Curve

Theorem

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

The Same Drawing for $y = \cos(x)$



Notes

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$\cos(0) = 1$

$\cos(\pm\frac{\pi}{2}) = 0$

$\cos(\pm\pi) = -1$

$\cos(\pm\frac{3\pi}{2}) = 0$

$\cos(\pm 2\pi) = 1$

$$\frac{d}{dx}\left[\csc(x)\right] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right]$$

$$= \frac{\frac{d}{dx}[1]\sin(x) - 1\frac{d}{dx}[\sin(x)]}{[\sin(x)]^2}$$

$$= \frac{0\cdot\sin(x) - 1\cdot\cos(x)}{[\sin(x)]^2}$$

$$= -\frac{\cos(x)}{\sin^2(x)} = \frac{-\cos(x)}{\sin(x)}\cdot\frac{1}{\sin(x)}$$

What Can the Derivative Rules Do?

Question

Which of the following derivatives can we NOT compute? (Assuming the derivative rules we have so far.)

1. $f(x) = x^3 + 2x + 1$ ✓
2. $g(x) = \sqrt{2x+3}$ → We need more tools.
3. $h(x) = \frac{x}{x^2+1}$ ✓
4. $i(x) = x\sin(x)$ ✓

Notes