Week 9: Riemann Sums

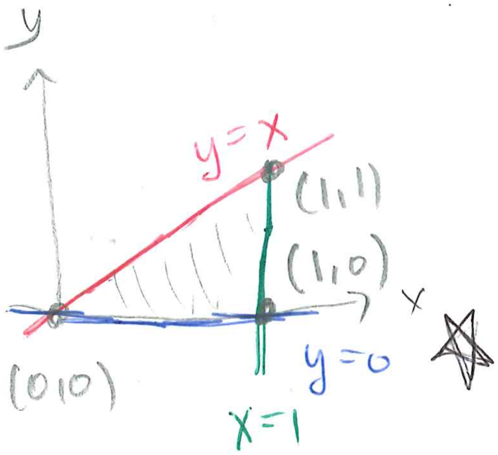
Remark: A Lot of Machinery

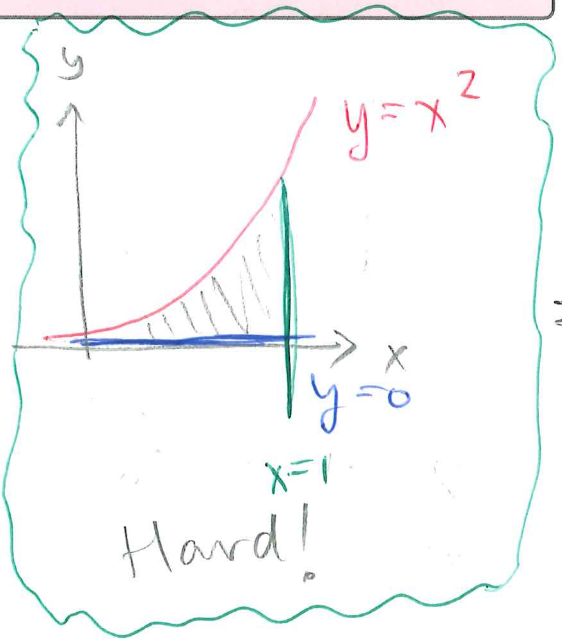
This week we begin our study of the area bounded by curves. The main tool we'll develop is the theory of Riemann sums. This material is a *lot* more technical than the course has been so far. If it is overwhelming remember: people can learn this stuff. You can learn it to. You can always ask for help.

The plan is to calculate an area in a "machinery free" way and then build up the theory progressively.

Example: A Highschool Problem and A Hard Problem

✓ *Highschool*: Calculate the area of the triangle bounded by the lines $y = 0$, $y = x$, and $x = 1$.
Hard: Find the area bounded by $y = 0$, $y = x^2$, and $x = 1$. (Archimedes is famous for solving this.)





⇒

$$A = \frac{1}{2} bh$$

$$= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

Easy.

The goal is to build up machinery to handle both these problems.

Setup the area bounded by $y=0$, $x=1$, ad $\boxed{y=x^2}$ as a Riemann sum with right-end points. and $\Delta x_K = \frac{1}{N}$.

The Riemann sum is:

$$\sum_{K=0}^{N} f(x_K^*)\,\Delta x_K$$

$$= \sum_{K=0}^{N} f(x_K^*)\,\frac{1}{N} \quad \# \Delta x_K = \frac{1}{N}$$

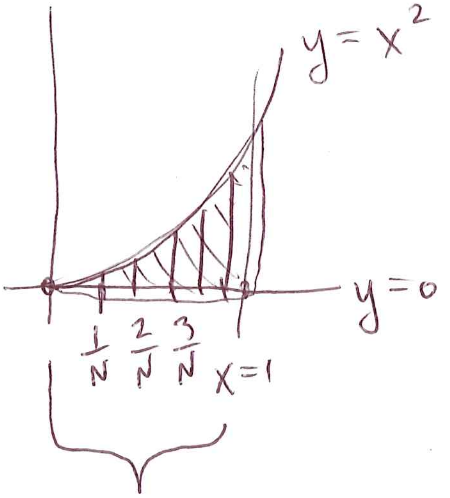
$$= \sum_{K=0}^{N} f(x_{K+1})\,\frac{1}{N} \quad \# x_K^* = x_{K+1}$$

$$= \sum_{K=0}^{N} f\left(\frac{K+1}{N}\right)\frac{1}{N} \quad \# x_K = \left(\frac{K}{N}\right)$$

$$= \sum_{K=0}^{N} \left(\frac{K+1}{N}\right)^2 \frac{1}{N} \quad \# f = x^2$$

$$= \frac{1}{N^3}\sum_{K=0}^{N}(K+1)^2$$

$$= \frac{1}{N^3}\sum_{k=0}^{N}\Big(\underset{\substack{\uparrow\\ \text{unknown}}}{\boxed{k^2}} + \underset{\substack{\uparrow\\ \text{known}}}{2k} + \underset{\uparrow}{1}\Big) \longrightarrow$$



$$\Delta x_K = \frac{1-0}{N} = \frac{1}{N}$$

The intervals are:

$$[x_K, \boxed{x_{K+1}}]$$

$$\parallel$$

$$x_K^*$$

We add k copies of Δx ad get:

$$x_K = K\left(\frac{1}{N}\right) = \frac{K}{N}$$

This reduces the area calculation to:

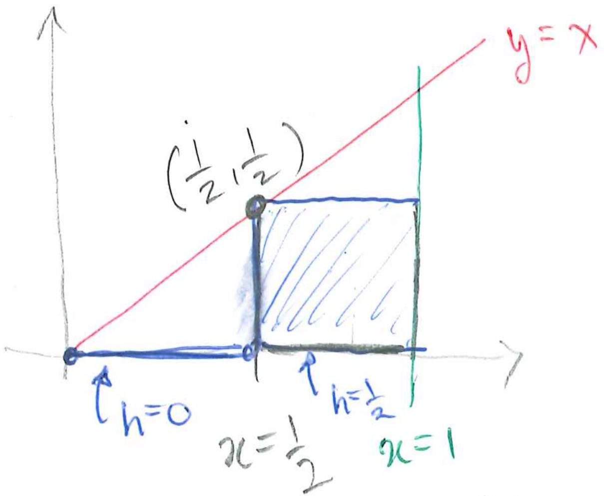
$$\sum_{k=0}^{N} k^2$$

Remark: Approximation!

The whole idea of Riemann sums rests on the idea of approximation. We take better and better approximations, until we get the actual area.

Example: Approximating The Triangle By Rectangles

The triangle T bounded by the lines $y = 0$, $y = x$, and $x = 1$ has base $[0, 1]$. Approximate the area of T by splitting the base in to two parts of equal length and erecting rectangles on bases. Write the **left end-point** approximation T_L and the **right end-point** approximation T_R separately.



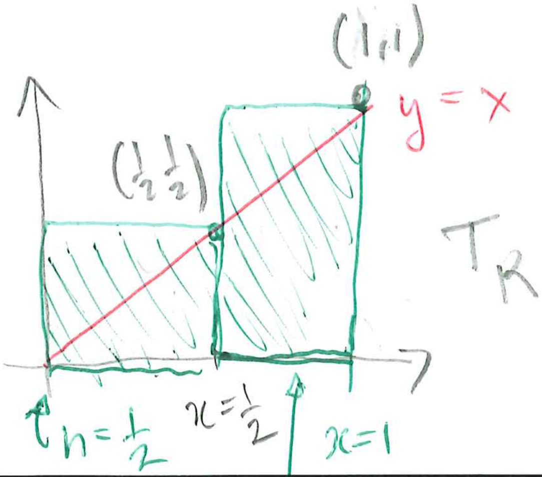
We split in to two parts of equal length:

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}$$

"delta"

We choose the left end-points for heights of rectangles ad get: Rect = hb

$$T_L = \boxed{0} \cdot \frac{1}{2} + \boxed{\frac{1}{2}} \cdot \frac{1}{2} = 0 + \frac{1}{4} = \frac{1}{4}.$$



Picking right end-points:

$$T_R = \boxed{\frac{1}{2}} \cdot \frac{1}{2} + \boxed{1} \cdot \frac{1}{2} = \frac{3}{4}.$$

The plan:

give a nice tool for calculating sums w. derivatives.

[0,1]

Activity: Try It Yourself (5 min)

Repeat the previous example but split the base [0, 1] of the triangle T in to three parts of equal length. Calculate T_L and T_R as before. What do you notice about the values T_L and T_R?

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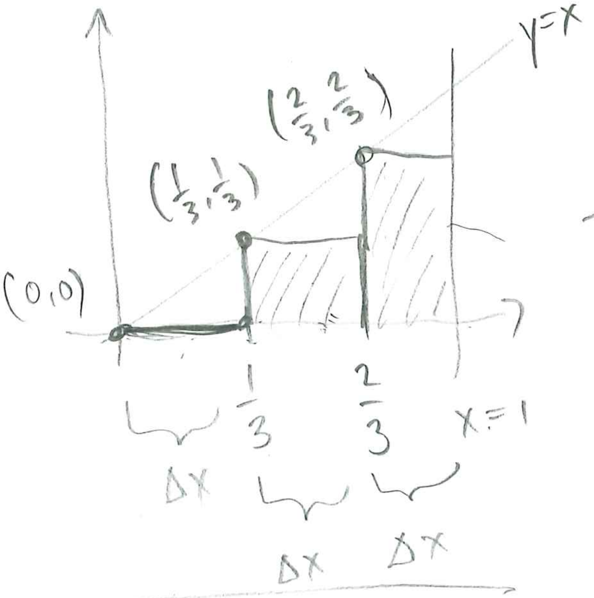


$$\Delta x = \frac{1-0}{3} = \frac{1}{3}$$

$$T_L = \left(\frac{1}{3}-0\right)0 + \left(\frac{2}{3}-\frac{1}{3}\right)\frac{1}{3} + \left(1-\frac{2}{3}\right)\frac{2}{3}$$

$$= \Delta x \cdot 0 + \Delta x \cdot \frac{1}{3} + \Delta x \cdot \frac{2}{3}$$

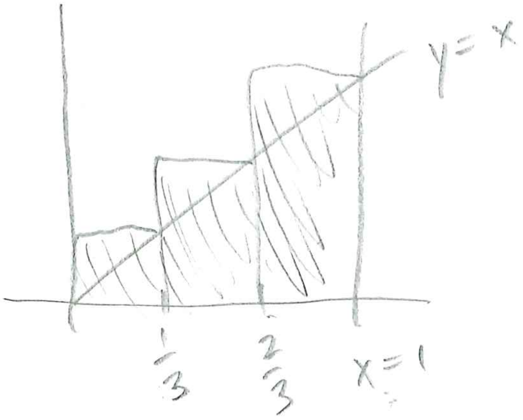
$$= \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} = \frac{3}{9} = \frac{1}{3}$$

$T_L <$ Area of Triangle $< T_R$

$$T_R = \Delta x\left(\frac{1}{3}\right) + \Delta x\left(\frac{2}{3}\right) + \Delta x(1)$$

$$= \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot 1$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{3}{9} = \frac{6}{9} = \frac{2}{3}$$



Remark: Why we need sequences.

Our approximations clearly depend on the number of pieces which we use to split up the triangle. We want a compact way to describe "the behaviour of T_L and T_R with n parts".

Awesome Facts

- $T_L + T_R = 1$

- $T_L = T_R + \Delta x$

Definition: Sequences

A **sequence** x_n is a list of real numbers with a value for each n in the naturals.
We also write $x_n = x(n)$ as a function of n. We call x_n a **term** of the sequence, and n is the **index** of x_n.

Example: Some Common Sequences

Compute the first five terms $n = 1, 2, 3, 4, 5$ of the following sequences:

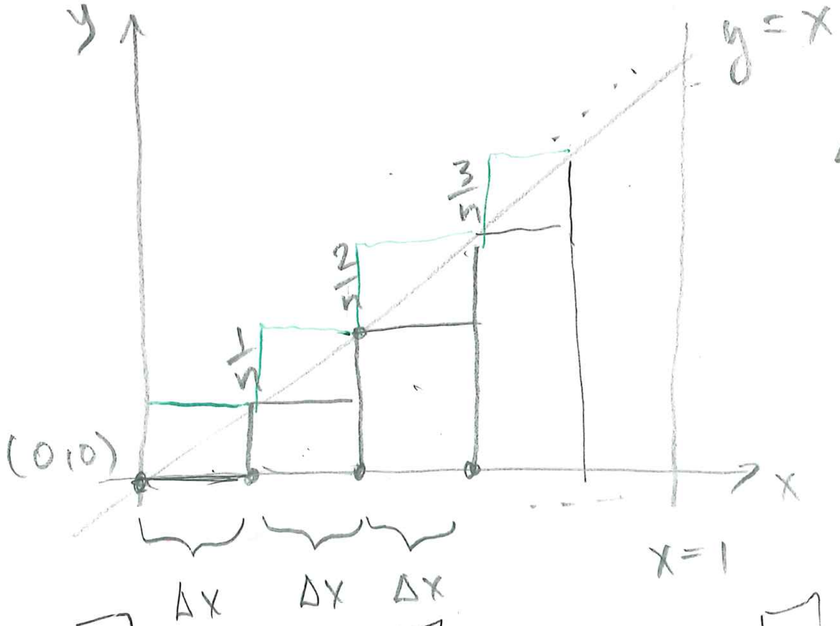
1. $x_n = n$
2. $x_n = 2^n$
3. $x_n = \frac{1}{n}$

n	$x_n = n$	$x_n = 2^n$	$x_n = \frac{1}{n}$
1	1	2	$1/1$
2	2	4	$1/2$
3	3	8	$1/3$
4	4	16	$1/4$
5	5	32	$1/5$

Goal: Get formulas for $T_L(n)$ ad $T_R(n)$.

Example: General Formulas for T_L and T_R

Suppose that we split the interval $[0, 1]$ in to n parts of equal length.
Write general formulas for $T_L(n)$ and $T_R(n)$.



$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$T_L(n) = \left(\frac{\boxed{0}}{n}\right)\Delta x + \left(\frac{\boxed{1}}{n}\right)\Delta x + \left(\frac{\boxed{2}}{n}\right)\Delta x + \left(\frac{\boxed{3}}{n}\right)\Delta x$$

$$+ \cdots + \left(\frac{\boxed{n-1}}{n}\right)\Delta x$$

$$T_R(n) = \left(\frac{1}{n}\right)\Delta x + \left(\frac{2}{n}\right)\Delta x + \left(\frac{3}{n}\right)\Delta x$$

$$+ \cdots\cdots + \left(\frac{n}{n}\right)\Delta x$$

$$= \left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right)\Delta x = \frac{1}{n}(1+2+\cdots+n)\Delta x$$

Remark: Why we need summation.

As we can see, our formulas for $T_L(n)$ and $T_R(n)$ involve a lot of "dot dot dots". We want a compact way to describe these summations so that we can use algebra and other tools to handle sequences.

Definition: Summation

Given a sequence x_n we can define its **sequence of partial sums** by:

$$S_N = x_1 + x_2 + \cdots + x_N = \sum_{k=1}^{N} x_k.$$

"Sigma"
~ "Sum"
~ "Summation"

The compact way of writing this involves **sigma notation**:

$$\begin{matrix} \text{upper} \longrightarrow \\ \\ \text{lower} \longrightarrow \end{matrix} \sum_{k=a}^{b} x_{K}$$

We call k the **dummy variable** or **index of summation**. The values $k = a$ and $k = b$ are the **lower** and **upper bound** respectively. *Note*: We may start the summation at $k = 1$ or another other value. Other common choices of dummy variable are i and n.

Example: Calculate Some Partial Sums

Calculate the following sums:

1. $\sum_{k=0}^{2}(2k+1) = \underbrace{(2\cdot 0+1)}_{K=0} + \underbrace{(2\cdot 1+1)}_{K=1} + \underbrace{(2\cdot 2+1)}_{K=2} = 1+3+5 = 9$
2. $\sum_{k=0}^{2} \frac{1}{2^{K}}$
3. $\sum_{k=3}^{6}(-1)^k = \underbrace{(-1)^3}_{K=3} + \underbrace{(-1)^4}_{K=4} + \underbrace{(-1)^5}_{K=5} + \underbrace{(-1)^6}_{K=6} = -1+1-1+1 = 0$

Example: Find a Formula

Evaluate the first four terms $N = 1, 2, 3, 4$ of the following and guess formulas for S_N:

1. $S_N = \sum_{k=0}^{N} \pi$
2. $S_N = \sum_{k=1}^{N} 42$

①

N	$S_N = \sum_{\boxed{k=0}}^{\boxed{N}} \pi \rightsquigarrow S_N = (N+1)\pi$
1	$S_1 = \sum_{k=0}^{1} \pi = \pi + \pi = 2\pi$
2	$S_2 = \sum_{k=0}^{2} \pi = \pi + \pi + \pi = 3\pi$
3	$S_3 = \sum_{k=0}^{3} \pi = \underbrace{\pi}_{k=0} + \underbrace{\pi}_{k=1} + \underbrace{\pi}_{k=2} + \underbrace{\pi}_{k=3} = 4\pi$

②

N	$S_N = \sum_{\boxed{k=1}}^{\boxed{N}} 42 \rightsquigarrow S_N = N \cdot 42$
1	$S_1 = \sum_{k=1}^{1} 42 = 42$
2	$S_2 = \sum_{k=1}^{2} 42 = 42 + 42 = 2 \cdot 42$
3	$S_3 = \sum_{k=1}^{3} 42 = 42 + 42 + 42 = 3 \cdot 42$

Example: Arithmetic Progressions

An **arithmetic progression** is $x_n = A + nB$. Find a formula for the partial sum $S_N = \sum_{k=0}^{N} x_k$.

For example,

$$x_1 = A + B$$
$$x_2 = A + 2B$$
$$x_3 = A + 3B$$

$$S_N = \sum_{K=0}^{N} x_K$$

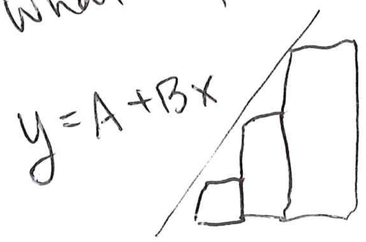
$$= \sum_{K=0}^{N} (A + KB)$$

$$= \sum_{K=0}^{N} A + \sum_{K=0}^{N} KB = \underbrace{\sum_{K=0}^{N} A}_{\text{Sum of Constants}} + B \underbrace{\sum_{K=0}^{N} K}_{\text{Sum of } K \text{ (Gauss)}}$$

$$\boxed{\sum_{K=0}^{N} K = \frac{N(N+1)}{2}}$$

$$= (N+1)A + B\,\frac{N(N+1)}{2}$$

What's the corresponding area calculation?



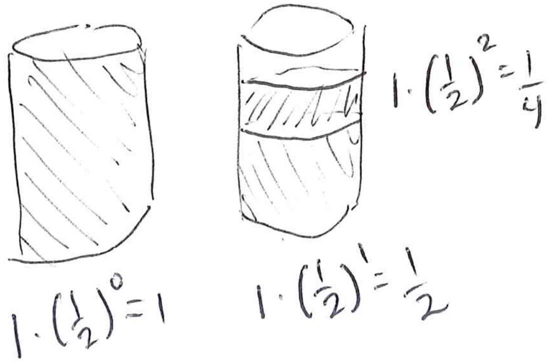
This would be used for calculating the area under $y = A + Bx$.

Example: Geometric Series

A **geometric series** is a series of the form:

$$S_N = \underbrace{ar^0 + ar^1 + \cdots + ar^N}_{} = \sum_{k=0}^{N} ar^k$$

Show that $S_N = a\dfrac{1-r^{N+1}}{1-r}$. (*Story*: The case $a = 1$ and $r = \dfrac{1}{2}$ has a silly story about pouring two beers.)



We have:

$$S_N = ar^0 + ar^1 + \dots + ar^N$$
$$= a(r^0 + r^1 + \dots + r^N)$$

We multiply both sides by $(1-r)$. This gives:

$$(1-r)S_N = a(1-r)(r^0 + \dots + r^N)$$
$$= a\begin{pmatrix} r^0 + r^1 + \dots + r^N \\ -r^1 - \quad - r^N - r^{N+1} \end{pmatrix}$$
$$= a(r^0 - r^{N+1})$$

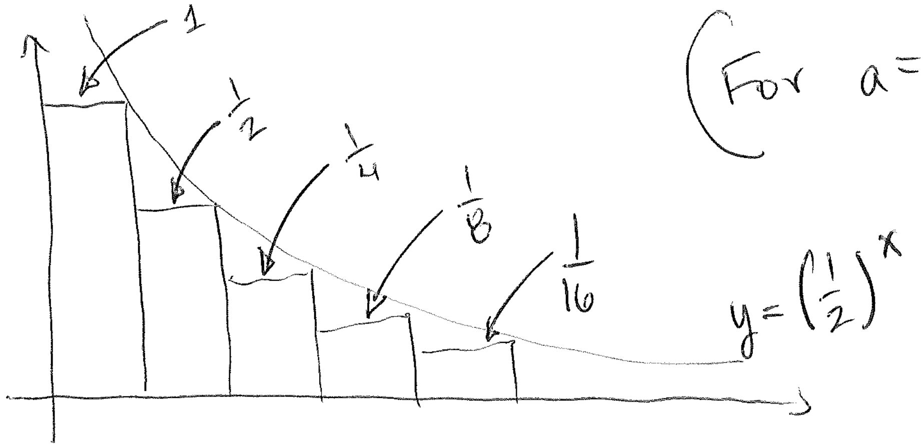
Divide both sides by $(1-r)$.

Therefore,

$$S_N = a\left(\frac{r^0 - r^{N+1}}{1-r}\right) = a\left(\frac{1-r^{N+1}}{1-r}\right)$$

What's the corresponding area calculation?

(For $a=1$ ad $r=\frac{1}{2}$.)



Example: Little Gauss's Sum

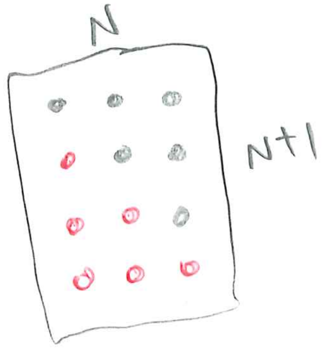
Find a formula for the following:

$$S_N = \sum_{k=1}^{N} k. = 1 + 2 + \cdots + N$$

Story: There is a famous story about the mathematician Gauss. When he was a little child, his teacher asked his whole class to add up the numbers from one to a hundred. In this notation, that question is "Calculate S_{100}." Gauss, the prodigy, instantly responded: 5050.

N	S_N		$\frac{N(N+1)}{2}$
1	1	$=1$	$1\cdot 2/2 = 1$
2	$1+2$	$=3$	
3	$1+2+3$	$=6$	
4	$1+2+3+4$	$=10$	$\frac{4(4+1)}{2} = 10$

Observe the following:



$$2S_N = 2\sum_{k=1}^{N} k \doteq \sum_{k=1}^{N} k + \sum_{k=1}^{N} k$$

$$= (1+N) + (2+(N-1)) + \cdots + ((N-1)+2)$$

$$= (1+N) + (1+N) + \cdots + (1+N)$$

$$= \sum_{k=1}^{N}(1+N) = N(1+N)$$

$$S_N = \frac{N(1+N)}{2}$$

Example: The Riemann Sum Area of T

Setup a general formula for $T_L(n)$ and $T_R(n)$. Take the limit as n goes to infinity.

Note: We ought to get $A = 1/2$ by highschool geometry.

Recall,

$$T_L(n) = \underbrace{\left(\frac{0}{n}\right)\Delta x}_{k=0} + \underbrace{\left(\frac{1}{n}\right)\Delta x}_{k=1} + \cdots + \underbrace{\left(\frac{n-1}{n}\right)\Delta x}_{k=n-1}$$

$$= \sum_{k=0}^{n-1}\left(\frac{k}{n}\right)\Delta x = \frac{1}{n}\Delta x\left(1+2+\cdots+(n-1)\right)$$

gauss

$$= \frac{1}{n}\cdot\frac{1-0}{n}\cdot\left(\frac{n(n-1)}{2}\right) = \frac{n^2-n}{2n^2} = \frac{1}{2}\left(1-\frac{1}{n}\right)$$

$$T_{R(n)} = \underbrace{\left(\frac{1}{n}\right)\Delta x}_{k=1} + \underbrace{\left(\frac{2}{n}\right)\Delta x}_{k=2} + \cdots + \underbrace{\left(\frac{n}{n}\right)\Delta x}_{k=n}$$

$$= \sum_{k=1}^{n}\left(\frac{k}{n}\right)\Delta x = \frac{1}{n}\Delta x \boxed{\sum_{k=1}^{n} k} \leftarrow$$

we have a formula for this one! (gauss.)

$$= \frac{1}{n}\Delta x\,\frac{n(n+1)}{2} = \frac{1}{n}\cdot\frac{1-0}{n}\cdot\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2n^2} = \frac{n^2+n}{2n^2} = \frac{n^2(1+\frac{1}{n})}{n^2(2)} = \frac{1+\frac{1}{n}}{2}$$

$T_R(n) \longrightarrow$ Area of Triangle $= \frac{1}{2}$ as $n \to \infty$.

we have the following:

$$T_L(n) = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$ ad

$$T_R(n) = \frac{1}{2}.\left(1 + \frac{1}{n}\right).$$

This gives:

$$\begin{tikzcd}
T_L(n) \arrow[d] & < & \text{Area} & < & T_R(n) \arrow[d] \\
\frac{1}{2}^{-} & & & & \frac{1}{2}^{+}
\end{tikzcd}$$

$\lim_{n \to \infty} T_L(n) = \frac{1}{2}^{-}$ ad $\lim_{n \to \infty} T_R(n) = \frac{1}{2}^{+}$.

🏃 Activity: Class Discussion (5 min)

Look over our calculation of the area of T. Here are some questions to consider:

- What's the difference between $\boxed{T_L(n)}$ and $T_R(n)$?
- What were the basic ingredients of the calculation?
- What really depended on the function $y = x$?

(12:39)

Definition: Riemann Sums

The example of T has led us to develop the theory of Riemann sums. To be concrete, a **Riemann sum** is:

$$\text{“The signed area bounded by } \underline{y = f(x)} \text{ on } [a,b]\text{”} = \lim_{N\to\infty} \sum_{k=0}^{N} f(x_k^*)\Delta x_k$$

height base

This definition has a lot of sub-parts. We name them now:

- The **end-points** are a sequence x_k such that: $a = x_0 < x_1 < \cdots < x_N = b$.
- Δx_k is the **length** of the interval $[x_k, x_{k+1}]$. $(\Delta x_k = x_{k+1} - x_k)$
- x_k^* is a **sample point** in the interval $[x_k, x_{k+1}]$.

In our calculation of T_L we had:

$$[a,b] = [0,1] \Rightarrow x_0 = a \text{ and } x_N = b$$

we had: $\Delta x_k = \frac{1-0}{N} = \frac{1}{N}$ for all k.

Left endpoint:

For left end-points, we had: $\boxed{x_k^* = x_k = \frac{k}{N}}$

For the function $f(x) = x$ we get:

$$\sum_{k=0}^{N} f(x_k^*)\Delta x_k = \sum_{k=0}^{N} x_k^* \Delta x_k = \sum_{k=0}^{N} \left(\frac{k}{N}\right)\Delta x_k$$

$$= \sum_{k=0}^{N} \left(\frac{k}{N}\right)\left(\frac{1}{N}\right).$$

https://pgadey.ca/notes/advice-for-students/#know-the-definitions