# Week 11: Partial Fractions

## Remark: Division of Polynomials

This week, we're going to explore a technique called "Partial Fractions". This is not so much a technique of integration, but a means of simplifying integrands so that they become simpler. It builds on the notion of long division. We first generalize the notion of division with remainder of integers to polynomials.

## Theorem: Integer Long Division

For any rational number, we can write:

$$\frac{p}{q} = n + \frac{r}{q}$$

where $0 \leq r < q$.

## Example: Long Division with Remainder

Find the remainder when 1221 is divided by four. Write your answer in the format:

$$\frac{1221}{4} = n + \frac{r}{4}$$

where $0 \leq r < 4$.

$$\begin{array}{r|rrrr} & 3 & 0 & 5 & \\ \hline 4 & 1 & 2 & 2 & 1 \\ & 1 & 2 & & \\ \hline & 0 & 0 & 2 & 1 \\ & & & 2 & 0 \\ \hline & & & & 1 \end{array}$$

$\leftarrow$ Remainder: 1

This gives:

$$\frac{1221}{4} = 305 + \frac{1}{4}$$

12:55

## Activity: Which Method? (3 min)

Consider the following indefinite integrals. Which method would you try first?
(You don't need to evaluate the indefinite integral. Just pick a tool and say how you'd apply it.)

1. $\int x^2 + 2x \, dx$ ← direct.
2. $\int xe^{-x^2} \, dx$ ← substitution $u = -x^2$ and $du = -2x\,dx$
3. $\int x^2 + \cos(2x) \, dx$ ← direct + substitution $u = 2x$ ad $du = 2dx$.
4. $\int x^2 \cos(x) \, dx$ ← parts: $f' = \cos(x)$ ad $g = x^2$
5. $\int e^x \cos(x) \, dx$ ← parts: $f' = e^x$ and $g = \cos(x)$

We need two applications of parts.

Currently our methods are: direct, substitution, or parts.

We evaluate (4).

$f' = \cos x \Rightarrow f = \sin x$
$g = x^2 \Rightarrow g' = 2x$

$$\int x^2 \cos(x) dx = fg - \int fg' dx$$

$$= x^2 \sin(x) - \int (\sin x)(2x) dx$$

$$= x^2 \sin(x) - 2\int \boxed{x \sin(x)} dx$$

Do parts again!

$f' = \sin x \Rightarrow f = -\cos x$
$g = x \Rightarrow g' = 1$

$$= x^2 \sin(x) - 2\left[fg - \int fg' dx\right]$$

$$= x^2 \sin(x) - 2\left[-x\cos(x) - \int (-\cos x) \cdot 1 \, dx\right]$$

$$= x^2 \sin(x) + 2x\cos(x) - 2\int \cos x \, dx$$

Do direct integration!



$$= x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$$

We evaluate (3) $\int x^2 + \cos(2x)\,dx$

$$\int x^2 + \cos(2x)\,dx = \int x^2\,dx + \int \cos(2x)\,dx$$

$$= \frac{1}{3}x^3 + \int \cos(2x)\,dx$$

Let $u = 2x$
$du = 2\,dx$

$$= \frac{1}{3}x^3 + \frac{1}{2}\int \cos(\underbrace{2x}_{u})\,\underbrace{2\,dx}_{du} = \frac{1}{3}x^3 + \frac{1}{2}\int \cos(u)\,du$$

$$= \frac{1}{3}x^3 + \frac{1}{2}\sin(u) + C = \frac{1}{3}x^3 + \frac{1}{2}\sin(2x) + C.$$

---

We evaluate (2) $\int x e^{-x^2}\,dx$

Let $u = -x^2$
$du = -2x\,dx$

$$\int x e^{-x^2}\,dx = -\frac{1}{2}\int \underbrace{(-2x)}_{du}\, e^{\overbrace{-x^2}^{u}}\,dx = -\frac{1}{2}\int e^u\,du$$

$$= -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

? How do you know which parts to choose?

Practice ad experience.

If you pick the "wrong" order, degrees will increase or you'll need anti-derivatives that you don't have.

## Example: Long Division with Remainder

Find the remainder when 123 is divided by 11. Write your answer in the format:

$$\frac{123}{11} = n + \frac{r}{11}$$

where $0 \le r < 11$.

$$\begin{array}{r|lll} & & 1 & 1 \\ \hline 11 & 1 & 2 & 3 \\ & 1 & 1 & \\ \hline & & 1 & 3 \\ & & 1 & 1 \\ \hline & & & 2 \end{array}$$

2 ← Remainder

This gives:

$$\frac{123}{11} = 11 + \frac{2}{11}$$

This remainder is ALWAYS less than the denominator.

Numerator bigger than denominator

Numerator smaller than denominator.

## Remark: The Big Leap to Polynomials

We now make the big leap to polynomial long division. The idea here is that we treat each degree as a "digit". We then proceed to handle the division "digit" by "digit".

## Example: Polynomial Division

Divide the polynomial $x^2 + 2x + 1$ by $x + 1$ and express your answer in the format:

$$\frac{x^2 + 2x + 1}{x + 1} = p(x) + \frac{r(x)}{x + 1}$$

where $0 \le \deg(r) < \deg(x + 1)$.

$$\begin{array}{r|lll} & x & +1 & \\ \hline x+1 & x^2 & +2x & +1 \\ & x^2 & +x & \\ \hline & & x & +1 \\ & & x & +1 \\ \hline & & & 0 \end{array}$$

0 ← Remainder

This gives:

$$\frac{x^2+2x+1}{x+1} = x+1 + \frac{0}{x+1}$$

$$= x+1$$

This is not super-duper surprising

$$\frac{x^2+2x+1}{x+1} = \frac{(x+1)^2}{x+1}$$

$$= x+1$$

## Example: Polynomial Division

Divide the polynomial $x^3 - 1$ by $x + 3$ and express you answer in the format:

$$\frac{x^3 - 1}{x + 3} = p(x) + \frac{r(x)}{x + 3}$$

where $0 \leq \deg(r) < \deg(x + 3)$.

$$x^2 - 3x + 9$$

$$x + 3 \,\big)\, x^3 + 0x^2 + 0x - 1$$

$$x^3 + 3x^2$$

$$-3x^2 + 0x$$

$$-3x^2 - 9x$$

$$9x - 1$$

$$9x + 27$$

$-28$ ← Remainder

This gives:

$$\frac{x^3 - 1}{x + 3} = x^2 - 3x + 9 - \frac{28}{x + 3}$$

**Theorem: Polynomial Long Division**

For any rational function, we can write:

$$\frac{f(x)}{q(x)} = p(x) + \frac{r(x)}{q(x)}$$

where $\deg(r) < \deg(q)$. (This lets us simplify the integrand by reducing the degree.)

**Example: A Rational Function**

$$\int \frac{x^3 - 1}{x + 3} dx$$

*Note*: We've already calculated this polynomial long division.

We see $\deg(x^3 - 1) > \deg(x+3)$ and so we divide:

$$\frac{x^3-1}{x+3} = \underline{x^2 - 3x + 9} - \underline{\frac{28}{x+3}}$$

For our integral, this gives:

$$\int \frac{x^3-1}{x+3} dx = \int x^2 - 3x + 9 - \frac{28}{x+3} dx$$

$$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - \boxed{\int \frac{28}{x+3} dx}$$ # Power rule

# $\int \frac{1}{x} dx = \ln|x|$

$$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - \boxed{28 \ln|x+3|} + C.$$

$$\int \frac{28}{x+3} dx = 28 \int \frac{1}{x+3} dx = 28 \ln|x+3| \leftarrow \int \frac{1}{x} dx = \ln|x| + C.$$

**Example: A Rational Function**

$$\int \frac{x^2+2x+1}{x+3}dx$$

We notice $\deg(x^2+2x+1) > \deg(x+3)$ so we divide.

$$\begin{array}{r|l} & \quad x - 1 \\ x+3 & x^2 + 2x + 1 \\ & x^2 + 3x \\ \hline & \quad -x + 1 \\ & \quad -x - 3 \\ \hline & \qquad 4 \end{array}$$

4 ← Remainder.

This gives:

$$\int \frac{x^2+2x+1}{x+3}dx = \int x - 1 + \frac{4}{x+3}dx$$

$$= \frac{1}{2}x^2 - x + 4\ln|x+3| + C.$$

## Remark: Partial Fractions

We now get to the mechanics of actual partial fractions decomposition. If a rational function $p(x)/q(x)$ has $\deg(p) < \deg(q)$ then we might be able to re-write it in a simpler format. We give a few examples before beginning the general theory. We also illustrate two distinct methods of finding the unknown coefficients.

## Example: A Surprising Equality

Use the equality

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$$

to compute the integral

$$\int \frac{1}{1-x^2}dx.$$

We have $\deg(1) < \deg(1-x^2)$. Good!

$$\int \frac{1}{1-x^2}dx = \int \frac{1}{2}\cdot\frac{1}{1-x} + \frac{1}{2}\cdot\frac{1}{1+x}\,dx$$

$$= \frac{1}{2}\int\frac{1}{1-x}dx + \frac{1}{2}\int\frac{1}{1+x}dx$$

$$= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + C.$$

Check: $\frac{\partial}{\partial x}\left[-\frac{1}{2}\ln|1-x|\right] = -\frac{1}{2}\frac{1}{1-x}(-1)$ # chain rule

$$= \frac{1}{2}\frac{1}{1-x}$$

**Example: Re-write The Polynomial as A Sum**

Write the rational function in the given format: $\frac{1}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$.
Method #1: Find the values $A$ and $B$ by setting up a linear system with two equations and two unknowns.

$$\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$= \frac{A(x+3) + B(x+2)}{(x+2)(x+3)}$$

the numerators must match!

$$= \frac{Ax + 3A + Bx + 2B}{(x+2)(x+3)}$$

$$= \frac{(A+B)x + (3A+2B)}{(x+2)(x+3)}$$

$3A + 2B = 1$

We get:

$$0x + 1 = 1 = (A+B)x + (3A+2B)$$

$0 = A+B$

We get:

$$\begin{cases} 0 = A+B \\ 1 = 3A+2B \end{cases}$$

Therefore, $(A,B) = (1,-1)$ and

$$\frac{1}{x^2+5x+6} = \frac{1}{x+2} - \frac{1}{x+3}$$

**Example: Re-write The Polynomial as A Sum**

Write the rational function in the given format: $\dfrac{1}{x^2-3x+2} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$.
Method #2: Find the values of $A$ and $B$ by subbing in specific values of $x$.

$$\frac{1}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$= \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

The numerators must match so we get:

$$1 = A(x-2) + B(x-1)$$

We plug in $x=2$ and get:

$$1 = A(2-2) + B(2-1) \Rightarrow 1 = B$$

We plug in $x=1$ and get:

$$1 = A(1-2) + B(1-1) \Rightarrow 1 = A(-1)$$

$$\Rightarrow -1 = A$$

We get:

$$\frac{1}{x^2-3x+2} = \frac{-1}{x-1} + \frac{1}{x-2}$$

12:25

## Activity: Try it! (5 min)

Write the rational function in the given format: $\frac{2x-1}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$.
How should we adapt Method #1 and #2 when there is a non-constant numerator?

$$\frac{2x-1}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

$$= \frac{A(x+5)+B(x-3)}{(x-3)(x+5)}$$

Match!

$$2x-1 = A(x+5)+B(x-3)$$

we plug in $x=3$ and get:

$$2\cdot 3 - 1 = A(3+5) + B(3-3)$$

$$5 = A\cdot 8 \Rightarrow A = \frac{5}{8}$$

We plug in $x=-5$ and get:

$$2(-5)-1 = A(-5+5)+B(-5-3)$$

$$-11 = B(-8) \Rightarrow B = \frac{11}{8}$$

Therefore

$$\frac{5/8}{x-3} + \frac{11/8}{x+5} = \frac{2x-1}{(x-3)(x+5)}$$

**Example: Three Roots**

Produce a partial fractions decomposition of

$$\frac{6+4x}{6+11x+6x^2+x^3}$$

given the factorization $6+11x+6x^2+x^3=(x+1)(x+2)(x+3)$.

We want:

$$\frac{6+4x}{6+11x+6x^2+x^3} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$= \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)}{(x+1)(x+2)(x+3)}$$

A ad C

Eliminate B and C simultaneously.

We need:

$$6+4x = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Eliminate A ad C.

* We plug in $x=-1$ and get:

$$6+4(-1) = A(-1+2)(-1+3) \Rightarrow 2 = A\cdot 1\cdot 2 \Rightarrow A=1$$

* we plug in $x=-2$ and get:

$$6+4(-2) = B(-2+1)(-2+3) \Rightarrow -2 = B(-1)(1) \Rightarrow B=2$$

* we plug in $x=-3$ and get:

$$6+4(-3) = C(-3+1)(-3+2) \Rightarrow -6 = C(-2)(-1) \Rightarrow C=-3$$



We get: $\frac{1}{x+1} + \frac{2}{x+2} - \frac{3}{x+3}$.

12:40

## 🏃 Activity: What Goes Wrong? (2 min)

Try to write the rational function in the given format: $\dfrac{x+4}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x+1}$.

What goes wrong? What is weird here?

$$\frac{x+4}{(x+1)^2} = \frac{A(x+1) + B(x+1)}{(x+1)(x+1)}$$

This gives:

$$x+4 = A(x+1) + B(x+1)$$

Oh no! We can't solve for A or B.

This issue will come up any time the denominator has a repeated root. (In this case $(x+1)^2$ ...)

**Example: A Decomposition with a Repeated Factor**

Write the rational function in the given format: $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$.

(!) The fix for repeated roots is to introduce a summand for each power of the root.

$$\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$$

$$= \frac{A(x+1)+B}{(x+1)^2}$$

Match!

we get: $x+4 = A(x+1)+B$

we plug in $x=-1$ and get:

$$(-1)+4 = A(-1+1)+B$$

$$3 = B$$

We plug in ANY number (except $x=-1$)

Let's take $x=1$:

$$1+4 = A(1+1)+3 \Rightarrow 5-3 = 2A$$



$$\Rightarrow 2 = 2A$$

$$\Rightarrow A = 1$$

12:52

## Activity: Try it! (5 min)

Write the rational function in the given format: $\frac{2x+3}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$.

$$\frac{2x+3}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{2}{x-2} + \frac{7}{(x-2)^2}$$

$$= \frac{A(x-2)}{(x-2)^2} + \frac{B}{(x-2)^2} = \frac{A(x-2)+B}{(x-2)^2}$$

We get:

$$2x+3 = A(x-2)+B$$

We plug in $x=2$ and get:

$$2\cdot 2+3 = A(2-2)+B \Rightarrow B=7$$

We plug in ANY $x$ except $x=2$ and get:

$x=0$ $\quad 2\cdot 0+3 = A(0-2)+B$

$$\Leftrightarrow 3 = -2A+7$$

$$\Leftrightarrow -4 = -2A$$

$$\Leftrightarrow A=2$$