Week 12: Area and Volume

Remark: Riemann Sums and Area

Recall that we developed the theory of Riemann sums to measure area.

height

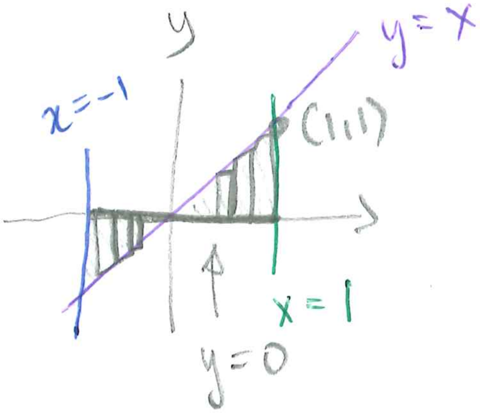
$$\text{“The signed area bounded by } y = f(x) \text{ on } [a, b]\text{”} = \lim_{N\to\infty} \sum_{k=0}^{N} f(x_k^*)\Delta x_k$$

base

This week, we'll look at applying this concept to measure area and volumes.

Example: What Does Riemann Really Measure?

Sketch the regions bounded by $y = x$, $x = -1$, $x = 1$, and $y = 0$.
Use a definite integral to measure this "area". Why is the answer surprising?



The area from $x=0$ to $x=1$:

$$\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_{x=0}^{x=1} = \frac{1}{2}1^2 - \frac{1}{2}\cdot 0^2 = \frac{1}{2}$$

The area from $x=-1$ to $x=0$.

$$\int_{-1}^0 x\,dx = \left[\frac{1}{2}x^2\right]_{x=-1}^{x=0} = \frac{1}{2}\cdot 0^2 - \frac{1}{2}(-1)^2 = -\frac{1}{2}$$

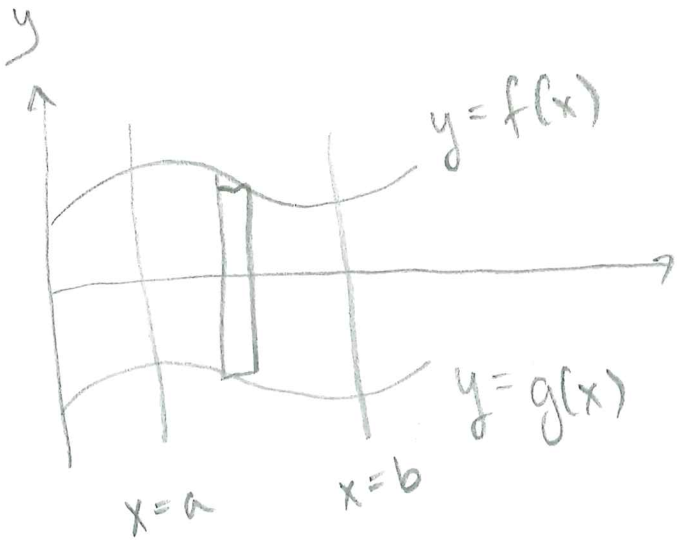
The total "area" is: $A = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0$

This is strange! We expect non-zero area!

Theorem: Area Between Curves

If $g(x) \leq f(x)$ on the interval $[a, b]$ then the area bounded by $x = a$, $x = b$, $y = f(x)$ and $y = g(x)$ is:

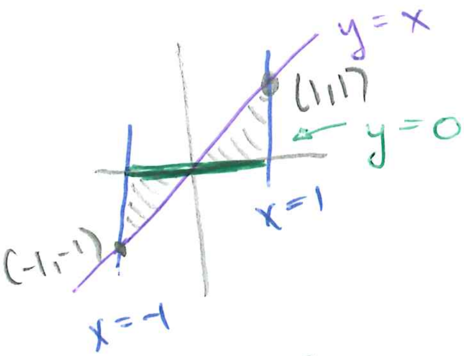
$$A = \int_a^b f(x) - g(x)dx$$



Height of rect $= f(x) - g(x) \geq 0$

Example: The Unsigned Area

Use highschool geometry to measure the true area of the region in the previous example.
Setup and evaluate an integral to measure its area.



Highschool:

$A = \frac{1}{2}bh + \frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} + \frac{1}{2} = 1$

As an integral:

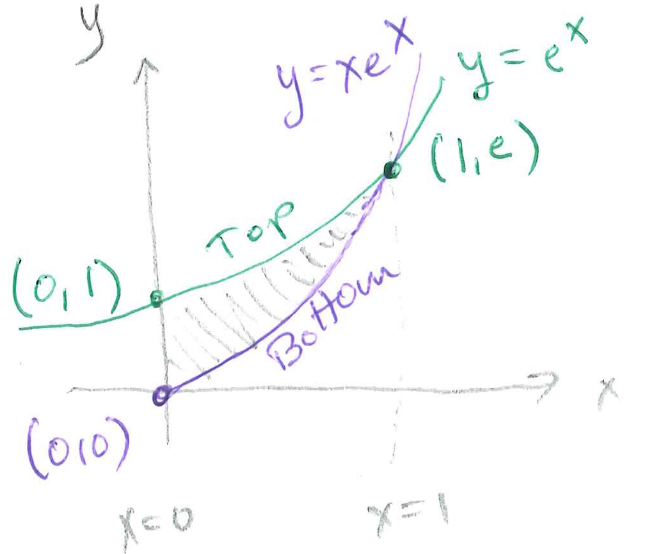
$$A = \int_{-1}^{0} 0 - x \, dx + \int_0^1 x - 0 \, dx = -\int_{-1}^{0} x \, dx + \int_0^1 x \, dx$$

$$= -\left(-\frac{1}{2}\right) + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

Yay! They match!

Example: Complicated Integrals / Top and Bottom Curves

Find the area bouned by $y = xe^x$, $y = e^x$, $x = 0$, and $x = 1$.



$$A = \int_0^1 \text{Top} - \text{Bottom}\, dx$$

$$= \int_0^1 e^x - xe^x\, dx$$

$$= \int_0^1 e^x dx - \int_0^1 xe^x dx$$

$$= \left[e^x\right]_{x=0}^{x=1} - \int_0^1 xe^x dx \leftarrow \text{Parts here!}$$

$f = x \quad f' = 1$
$g' = e^x \Rightarrow g = e^x$

$$= \left[e^x\right]_0^1 - \left[fg - \int f'g\right]_0^1$$

$$= \left[e^x\right]_0^1 - \left[xe^x - \int e^x dx\right]_0^1$$

$$= \left[e^x\right]_0^1 - \left[xe^x - e^x\right]_0^1$$

$$= e^1 - e^0 - \left[(1e^1 - e^1) - (0e^0 - e^0)\right]$$

$$= e^1 - 1 - e^1 + e^1 - 1$$

$$= e^1 - 2$$

$$= e - 2$$

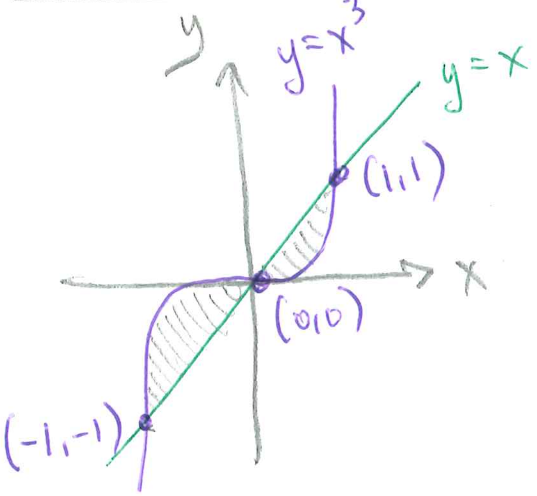
Remark: Curves Which Cross

It is inconvenient that we need $g(x) \leq f(x)$ over the whole interval $[a, b]$ because curves tend to cross each other. We can get rid of this restriction by considering:

$$A = \int_a^b |f(x) - g(x)| dx$$

Example: A Pair of Curves Which Cross

Find the unsigned area bounded by $y = x$ and $y = x^3$.



$$A = \int_{-1}^{1} |x - x^3| \, dx$$

To evaluate this we need to somehow get rid of the $|\cdot|$.

We need to determine when $\boxed{x - x^3 > 0}$ and $x - x^3 < 0$

$x^3 = x$

$\Leftrightarrow x^3 - x = 0$

$\Leftrightarrow x(x^2 - 1) = 0$

$\Leftrightarrow x(x-1)(x+1) = 0$

If $x - x^3 > 0$ then $x(x^2-1) > 0$.

We need: $\boxed{x > 0 \text{ and } x^2 - 1 > 0} \Leftrightarrow x > 1$

OR: $\boxed{x < 0 \text{ and } x^2 - 1 < 0} \Leftrightarrow -1 < x < 0$

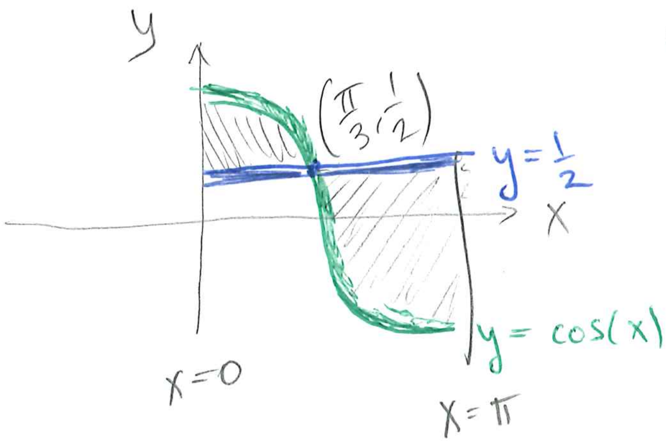
We get $x - x^3 > 0 \Leftrightarrow -1 < x < 0$.

This means we calculate:

$$A = \int_{-1}^{0} x - x^3 \, dx + \int_0^1 x^3 - x \, dx$$

Example: More Curves Which Cross

Find the unsigned area bounded by $y = \cos(x)$ and $y = \frac{1}{2}$ over $0 \le x \le \pi$.



We need $\cos(x) = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{3}$

$$A = \int_0^{\boxed{\frac{\pi}{3}}} \text{Top} - \text{Bottom}\, dx$$

$$+ \int_{\boxed{\frac{\pi}{3}}}^{\pi} \text{Top} - \text{Bottom}\, dx$$

$$= \int_0^{\boxed{\frac{\pi}{3}}} \cos(x) - \frac{1}{2}\, dx + \int_{\boxed{\frac{\pi}{3}}}^{\pi} \frac{1}{2} - \cos(x)\, dx$$

$$= \left[\sin(x) - \frac{1}{2}x\right]_0^{\boxed{\frac{\pi}{3}}} + \left[\frac{1}{2}x - \sin(x)\right]_{\boxed{\frac{\pi}{3}}}^{\pi}$$

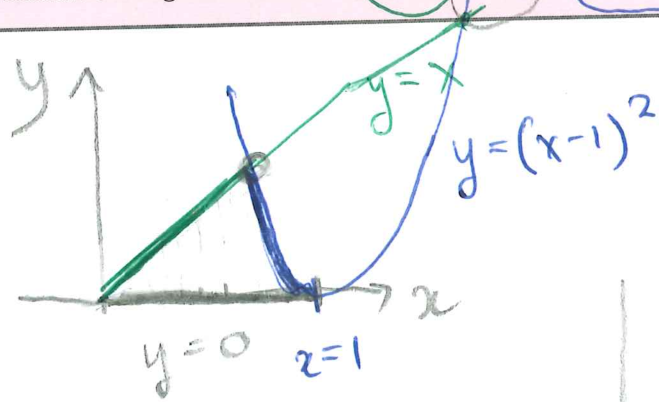
$$= \sin\left(\frac{\pi}{3}\right) - \frac{1}{2}\cdot\frac{\pi}{3} + \left[\frac{1}{2}\pi - \sin(\pi)\right] - \left[\frac{1}{2}\cdot\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\pi}{2} - 0 - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

$$= \sqrt{3} - \frac{\pi}{3} + \frac{\pi}{2} = \sqrt{3} - \frac{\pi}{6}.$$

Example: Complicated Regions

Sketch the region bounded by $y = x$, $y = 0$, and $y = (x-1)^2$. Find the area of this region.



$$A = \int_0^{\frac{3-\sqrt{5}}{2}} \text{Top} - \text{Bottom}\,dx + \int_{\frac{3-\sqrt{5}}{2}}^{1} \text{Top} - \text{Bottom}\,dx$$

We solve: $x = (x-1)^2$.

$x^2 - 2x + 1 = x$

$\Leftrightarrow x^2 - 3x + 1 = 0.$

$$x = \frac{3 \pm \sqrt{3^2 - 4\cdot 1\cdot 1}}{2\cdot 1} = \frac{3 \pm \sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$$

$$= \int_0^{\frac{3-\sqrt{5}}{2}} x\,dx + \int_{\frac{3-\sqrt{5}}{2}}^{1} (x-1)^2\,dx$$

$$= \left[\tfrac{1}{2}x^2\right]_0^{\frac{3-\sqrt{5}}{2}} + \left[\tfrac{1}{3}(x-1)^3\right]_{\frac{3-\sqrt{5}}{2}}^{1}$$

If we pick $x = \frac{3+\sqrt{5}}{2} > \frac{3}{2} > 1.$

We need: $x = \frac{3-\sqrt{5}}{2}$ so that $0 < x < 1$.

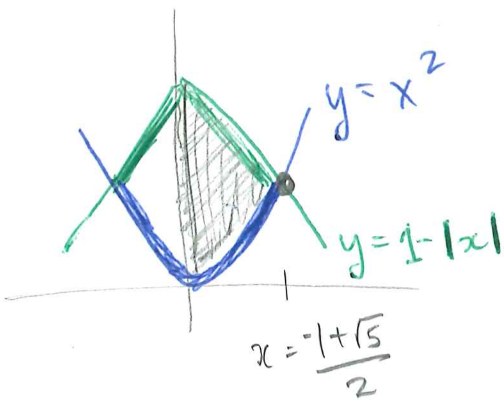
$$= \frac{1}{2}\left(\frac{3-\sqrt{5}}{2}\right)^2 + \frac{1}{3}(1-1)^3 - \frac{1}{3}\left(\frac{3-\sqrt{5}}{2} - 1\right)^3$$

$$= \frac{1}{2}\left(\frac{3-\sqrt{5}}{2}\right)^2 - \frac{1}{3}\left(\frac{3-\sqrt{5}}{2} - 1\right)^3$$

14:28

🏃 Activity: Try it! (5 min)

Use Desmos to sketch the region bounded by: [redacted], $y = 1 - |x|$, $y = x^2$.
Setup and evaluate an integral to compute this area.



Find point of intersection:

$$x^2 = 1 - |x| = 1 - x$$

$$\Leftrightarrow \quad x^2 + x - 1 = 0$$

$$\Leftrightarrow \quad x = \frac{-1 \pm \sqrt{1^2 - 4(-1)(1)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{5}}{2}$$

This is very symmetric so: $A = \underline{\underline{2}} \int_0^{\frac{-1+\sqrt{5}}{2}}$ Top − Bottom

We get the two because:

$$\int_{\frac{-1-\sqrt{5}}{2}}^{0} (1 - |x|) - x^2 \, dx = \int_0^{\frac{-1+\sqrt{5}}{2}} (1 - |x|) - x^2 \, dx$$

$$A = 2\int_0^{\frac{-1+\sqrt{5}}{2}} (1 - x) - x^2 \, dx = 2\left[x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^{\frac{-1+\sqrt{5}}{2}}$$

$$= 2\left(\frac{-1+\sqrt{5}}{2}\right) - \left(\frac{-1+\sqrt{5}}{2}\right)^2 - \frac{2}{3}\left(\frac{-1+\sqrt{5}}{2}\right)^3$$

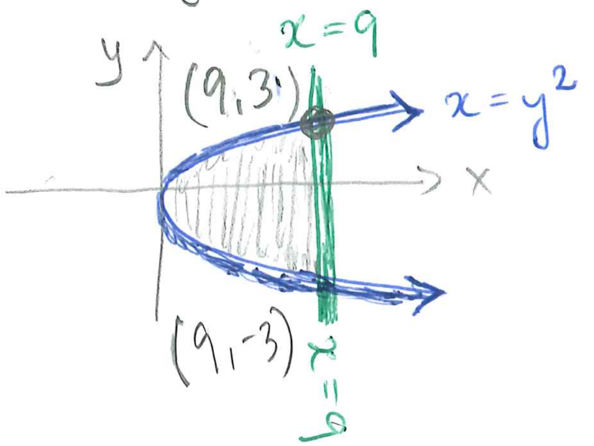
Remark: Area with Respect to y.

An notable property of area is that is doesn't "care about" how the region is bounded. It doesn't "care about" $y = f(x)$ versus $x = g(y)$. The region is bounded and has an area independent of the functions.

Example: An Area With Respect to y.

Find the area bounded by $x = y^2$ and $x = 9$.

(!) It is sometimes more convenient to bound things as functions of y than as functions of x.



$$A = \int_{-3}^{3} \text{Top} - \text{Bottom} \, dy$$

$$= \int_{-3}^{3} 9 - y^2 \, dy$$

We need points of intersection:

$9 = y^2 \Leftrightarrow y = \pm 3$

$$= \left[9y - \frac{1}{3} y^3 \right]_{y=-3}^{y=3}$$

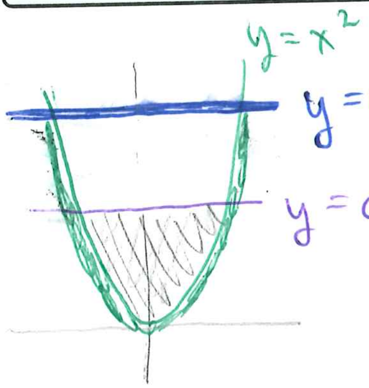
$$= \left(9 \cdot 3 - \frac{1}{3} \cdot 3^3\right) - \left(9(-3) - \frac{1}{3}(-3)^3\right)$$

$$= 27 - 9 - (-27 + 9) = 54 - 18 = 36$$

14 : 53

🏃 Activity: Lukas' Problem (5 min)

Find the line $y = c$ that cuts the region bounded by $y = x^2$ and $y = 1$ in half.

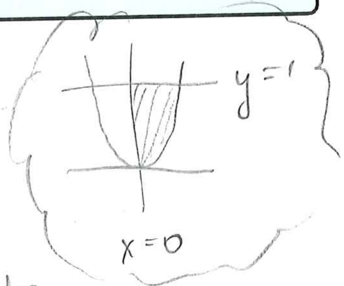


One approach:

① – Get total area.

② – Find $\frac{1}{2}$ total.

③ – Treat c as variable.



① $A = 2\int_0^1 \text{Top} - \text{Bottom}\, dx = 2\int_0^1 1 - x^2\, dx$

$= 2\left[x - \frac{1}{3}x^3\right]_0^1 = 2\left[1 - \frac{1}{3}\right] = \frac{4}{3}.$

② We want area bounded by $y=c$ ad $y=x^2$ to be:

$A = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}.$

③ $A(c) = 2\int_{\boxed{0}}^{\boxed{\sqrt{c}}} \text{Top} - \text{Bottom}\, dx = 2\int_{\boxed{0}}^{\boxed{\sqrt{c}}} c - x^2\, dx$

$= 2\left[cx - \frac{1}{3}x^3\right]_{\boxed{0}}^{\boxed{\sqrt{c}}}.$

We need the x-values of where $y=c$ meets $y=x^2$.

$= 2\cdot c^{\frac{3}{2}} - \frac{1}{3}c^{\frac{3}{2}} = \frac{5}{3}c^{\frac{3}{2}}$

Remark: Area and Volume

We've seen that we can get area as an integral.

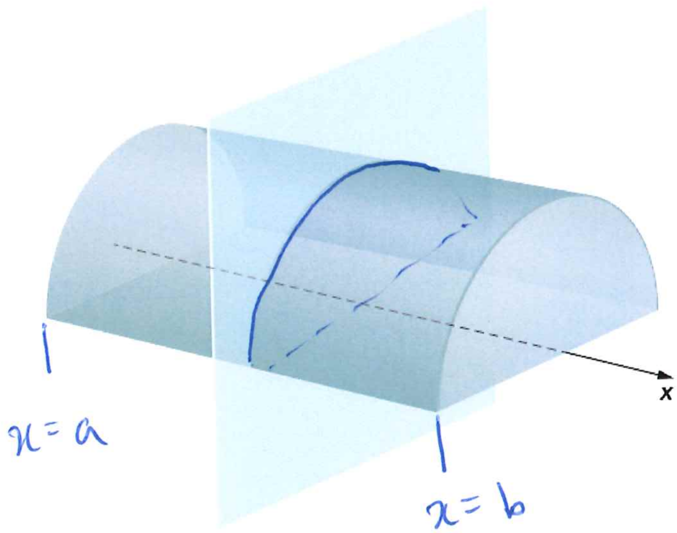
$$\text{“Area”} = \int_a^b \text{“Length”}\, dx$$

If we go one dimension higher, we can get:

$$\text{“Volume”} = \int_a^b \text{“Area”}\, dx$$

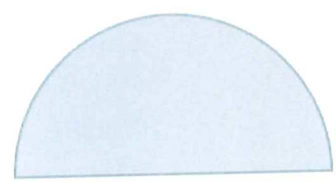
This is sometimes called Cavalieri's Principle after Galileo's student Bonaventura Cavalieri (1598-1647).

- Examine the solid and determine the shape of a cross-section of the solid.
  It is often helpful to draw a picture if one is not provided.
- Determine a formula for the area of the cross-section.
- Integrate the area formula over the appropriate interval to get the volume.



Three-dimensional cylinder

$A(x) = \text{Area}(x)$



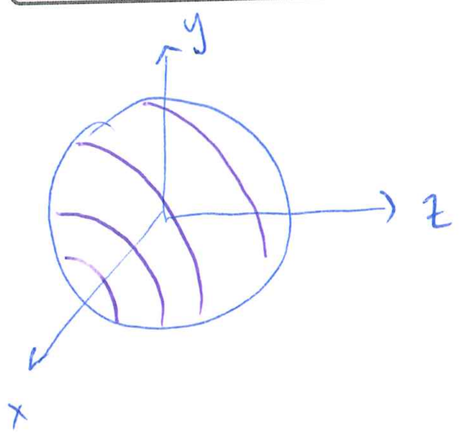
Two-dimensional cross section

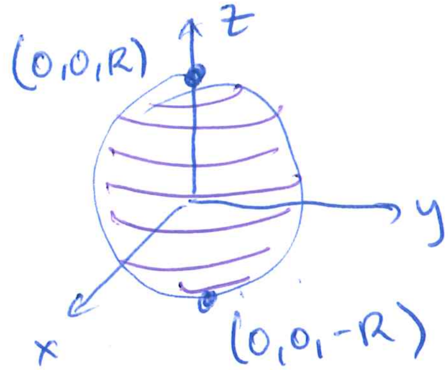
$$\text{Volume} = \int_a^b A(x)\, dx$$

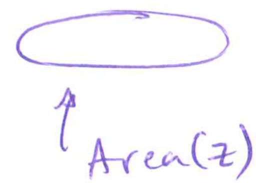
OpenStax §6.2 Determining Volume by Slicing

Example: Find The Volume of a Sphere

Find the volume of a sphere of radius $R > 0$ by slicing it parallel to the xy-plane and integrating Area(z).







We find a formula for Area(z)

$$x^2+y^2+z^2=R^2 \Longrightarrow x^2+y^2=R^2-z^2$$

The slice at height z is a circle of radius $\sqrt{R^2-z^2}$.

The area of the slice is $\text{Area}(z)=\pi r^2=\pi(R^2-z^2)$

We calculate volume:

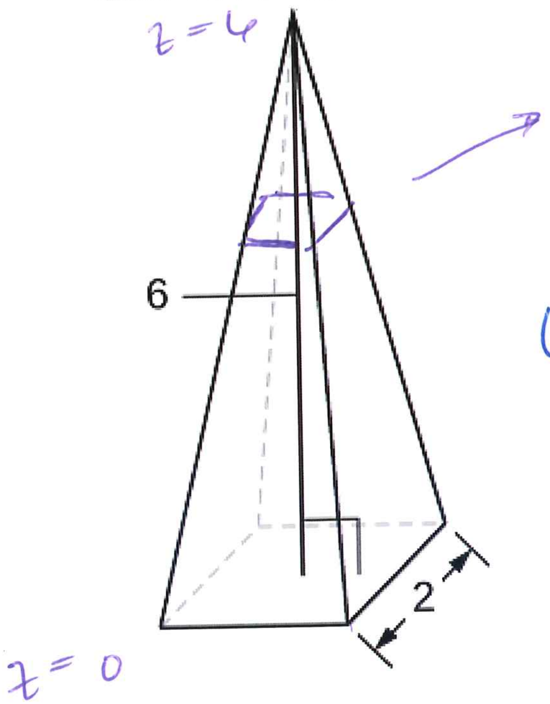
$$\text{Vol}=\int_{-R}^{R}\text{Area}(z)\,dz=\int_{-R}^{R}\pi(R^2-z^2)\,dz$$

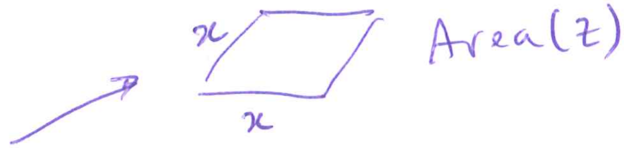
$$=\left[\pi\left(R^2z-\frac{1}{3}z^3\right)\right]_{-R}^{R}$$

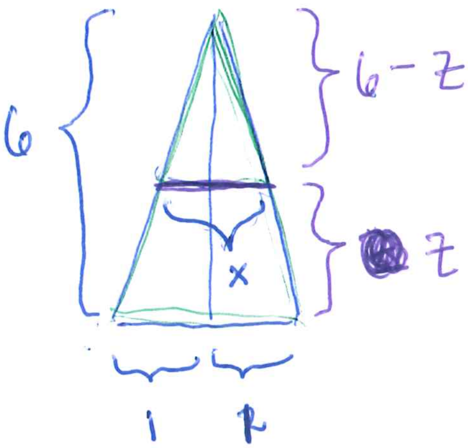
$$=\pi\left(R^3-\frac{1}{3}R^3\right)-\pi\left(-R^3+\frac{1}{3}R^3\right)=\frac{4}{3}\pi R^3$$

Example: A Pyramid (OS §6.2 Q63)

Find the volume of a pyramid of height six units and square base of sidelength two units.







By similar triangles, we get:

$$\frac{x}{2} = \frac{6-z}{6}$$

We get: $x = 2\left(\frac{6-z}{6}\right) = \frac{6-z}{3}$.

This gives:

$$\text{Area}(z) = x^2 = \left(\frac{6-z}{3}\right)^2$$

The total volume is therefore:

$$\text{Vol} = \int_0^6 \left(\frac{6-z}{3}\right)^2 dz = \left[-\frac{1}{3}\cdot 3\left(\frac{6-z}{3}\right)^3\right]_0^6$$

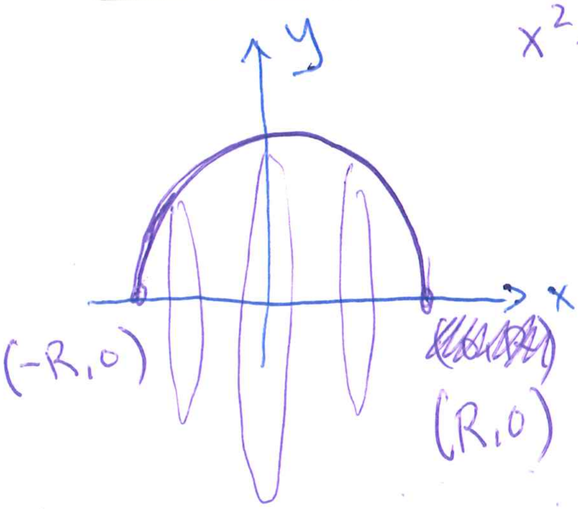
$$= -\frac{1}{3}\cdot 3\left(\frac{6-6}{3}\right)^3 + \frac{1}{3}\cdot 3\left(\frac{6-0}{3}\right)^3 = 2^3 = 8.$$

Definition: Volume of Revolution

We obtain a **solid of revolution** by revolving a curve $y = f(x)$ over $[a, b]$ around the x-axis. The volume of such a solid is a **volume of revolution**. (See the Desmos example below.)

Example: Find the Volume of a Sphere

Find the volume of the sphere of radius $R > 0$ as a volume of revolution.



$x^2 + y^2 = R^2 \Rightarrow y = \sqrt{R^2 - x^2} = f(x)$

We want to find the areas of the disks:

$\boxed{\text{Area}(x) = \pi r^2}$

$= \pi [f(x)]^2$

$= \pi \left[\sqrt{R^2 - x^2}\right]^2$

$= \pi (R^2 - x^2)$

Volumes of revolution have nice area functions.

The volume is therefore:

$$\text{Vol} = \int_{-R}^{R} \text{Area}(x)\,dx$$

$$= \int_{-R}^{R} \pi (R^2 - x^2)\,dx = \left[\pi\left(R^2 x - \tfrac{1}{3}x^3\right)\right]_{-R}^{R}$$

$$= \frac{4}{3}\pi R^3$$

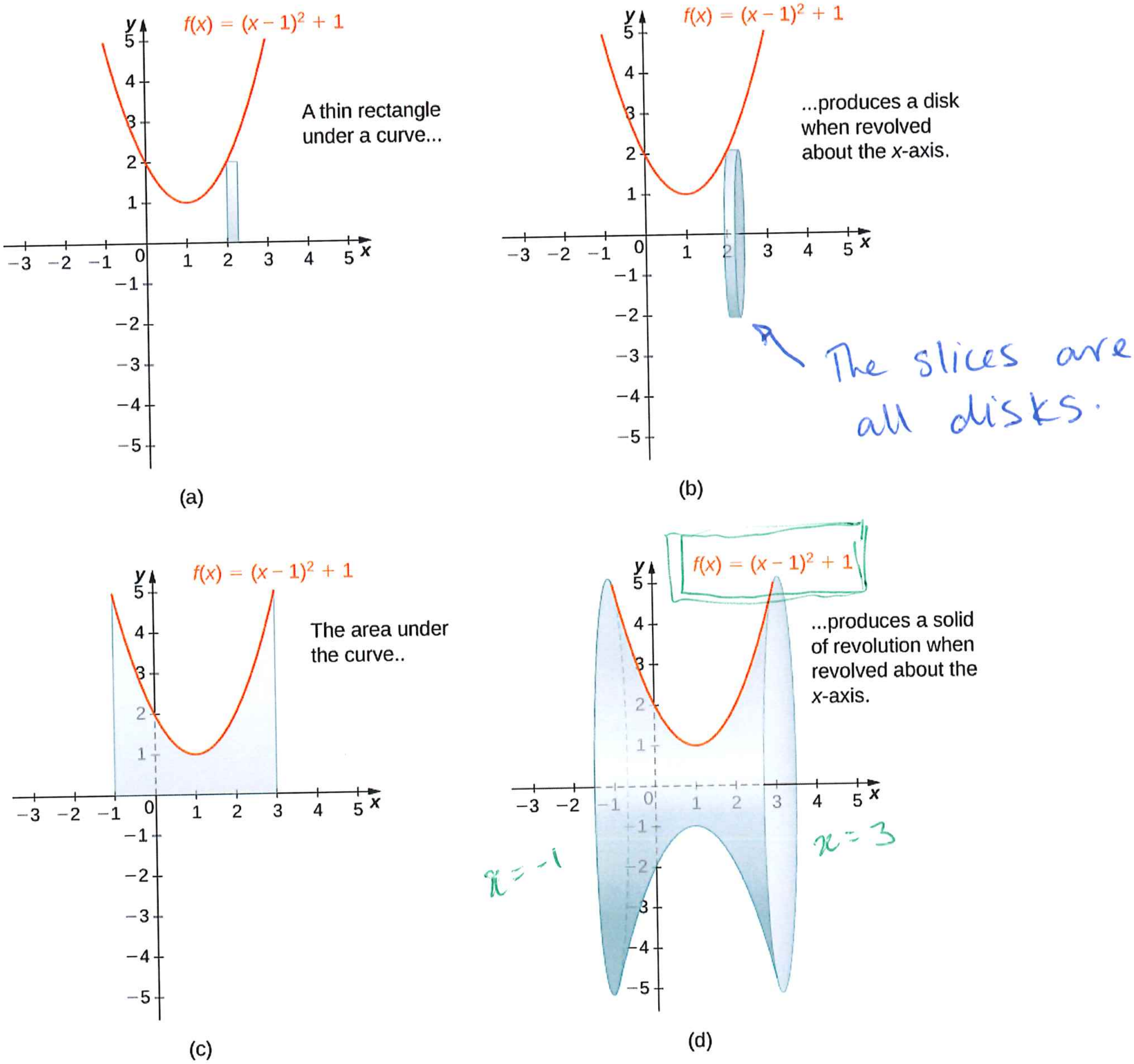
https://www.desmos.com/3d/1a7f86ac77

Theorem: The Disk Method

If $f(x)$ is non-negative on $[a,b]$ then the volume of revolution generated by $y = f(x)$ over $[a,b]$ is:

$$V = \int_a^b \text{Area}(x)\ dx = \int_a^b \pi[f(x)]^2 dx$$

OpenStax §6.2 Determining Volume by Slicing



Example: Applying the Disk Method

Find the volume of revolution of $y = (x-1)^2 + 1$ on the interval $[-1, 3]$.
(This is the example illustrated by the previous graphic.)

$$\text{vol} = \int_{-1}^{3} \text{Area}(x)\,dx = \int_{-1}^{3} \pi\,[f(x)]^2\,dx$$

$$= \int_{-1}^{3} \pi \left[(x-1)^2 + 1 \right]^2 dx$$

$$= \pi \int_{-1}^{3} \left[x^2 - 2x + 1 + 1 \right]^2 dx = \pi \int_{-1}^{3} \left[x^2 - 2x + 2 \right]^2 dx$$

$$= \pi \int_{-1}^{3} x^4 - 2x^3 + 2x^2 - 2x^3 + 4x^2 - 4x + 2x^2 - 4x + 4\,dx$$

$$= \pi \int_{-1}^{3} x^4 - 4x^3 + 8x^2 - 8x + 4\,dx$$

$$= \pi \left[\frac{1}{5}x^5 - x^4 + \frac{8}{3}x^3 - 4x^2 + 4x \right]_{-1}^{3}$$

$$= \pi \left(\frac{1}{5} 3^5 - 3^4 + \frac{8}{3} 3^3 - 4 \cdot 3^2 + 4 \cdot 3 \right)$$

$$- \pi \left(\frac{1}{5}(-1)^5 - (-1)^4 + \frac{8}{3}(-1) - 4(-1) + 4(-1) \right)$$

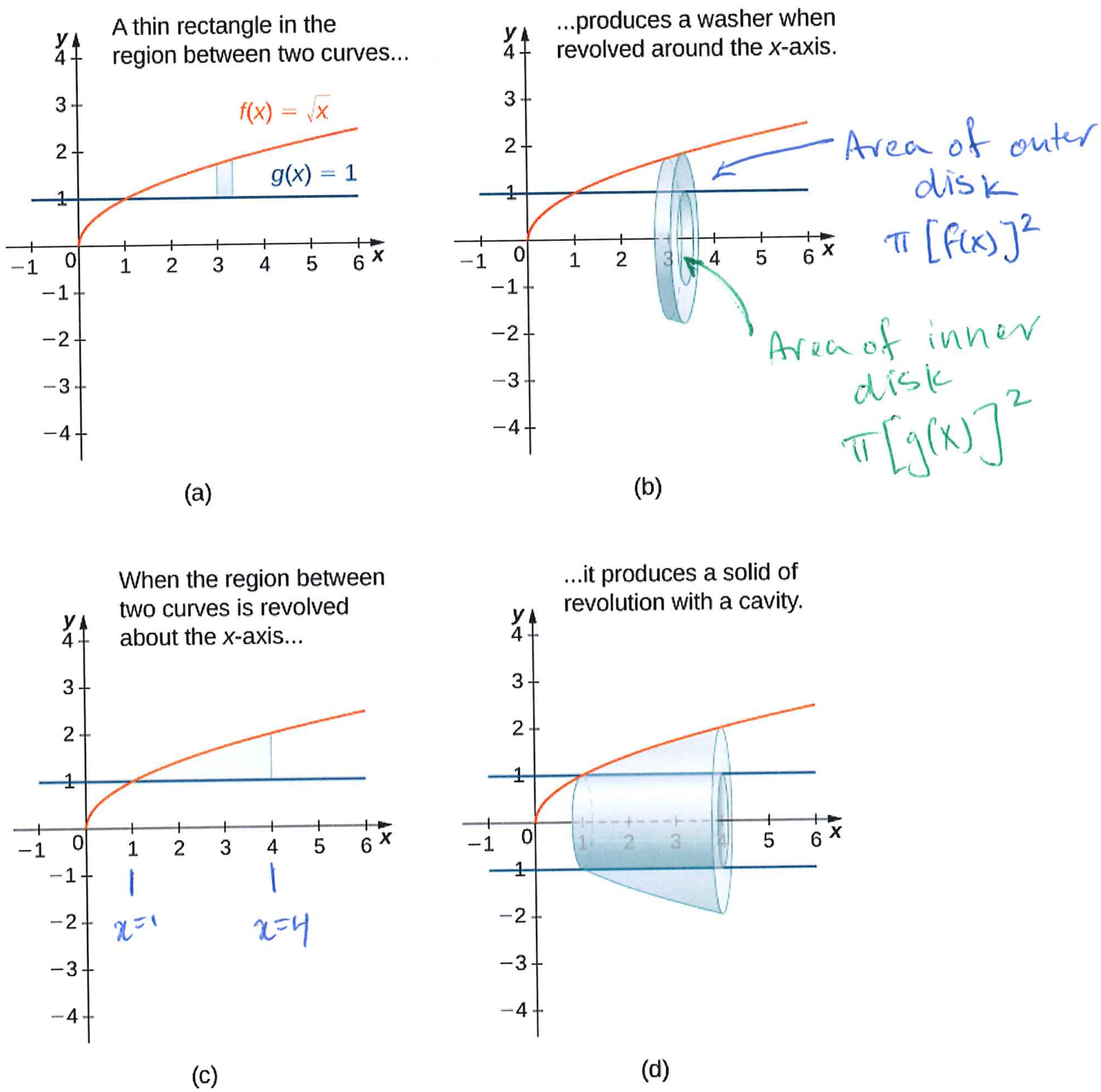
Calculator!

Theorem: The Washer Method

If $g(x) \leq f(x)$ are both non-negative on $[a, b]$ then the volume of revolution generated the region bounded by them over $[a, b]$ is:

$$V = \int_a^b \text{Area}(x)\ dx = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2\right) dx$$

OpenStax §6.2 Determining Volume by Slicing

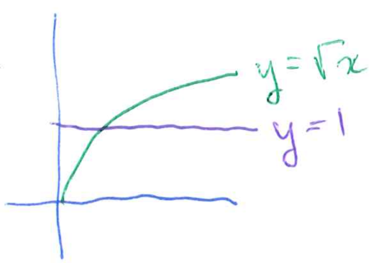


Example: Applying the Washer Method

Find the volume of revolution of $y = \sqrt{x}$ and $y = 1$ on the interval $[1, 4]$.
(This is the example illustrated by the previous graphic.)

$$\text{Vol} = \int_1^4 \text{Area}(x)\,dx = \int_1^4 \pi\left([f(x)]^2 - [g(x)]^2\right)dx$$

$$= \int_1^4 \pi\left([\sqrt{x}]^2 - [1]^2\right)dx$$



$$= \int_1^4 \pi(x-1)\,dx = \left[\pi\left(\tfrac{1}{2}x^2 - x\right)\right]_1^4$$

$$= \pi\left(\tfrac{1}{2}\cdot 4^2 - 4\right) - \pi\left(\tfrac{1}{2}\cdot 1^2 - 1\right)$$

$$= \pi(8-4) - \pi\left(-\tfrac{1}{2}\right) = 4\pi + \tfrac{1}{2}\pi = \tfrac{9}{2}\pi$$