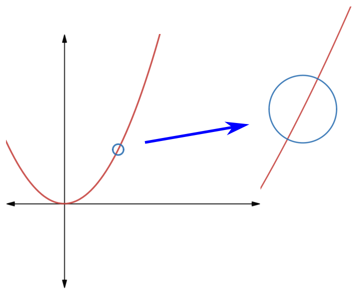
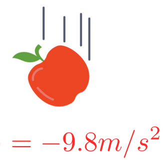
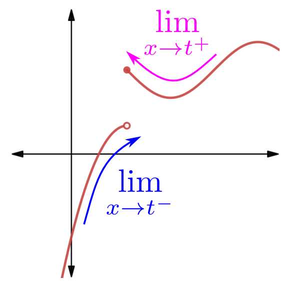
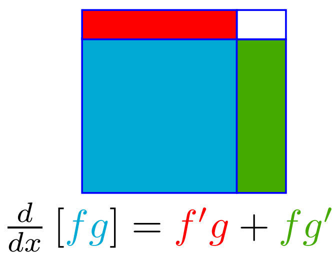
Week 1: Welcome to the Course and Introduction to Functions

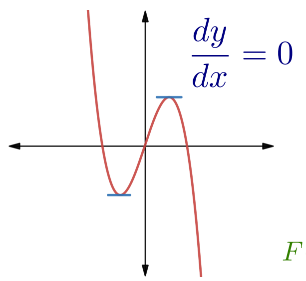


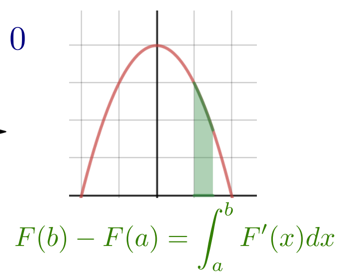


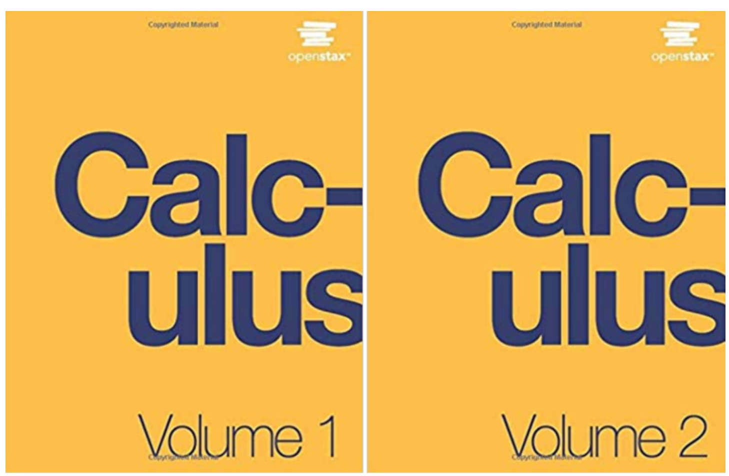


MAT A29
Calculus I for the Life Sciences









The Grading Scheme

Tests	$10\% + 20\% =$	30%
Exam	$1 \times 40\% =$	40%
Assignments	$(6-1) \times 3\% =$	15%
Quizzes	$(6-1) \times 3\% =$	15%

Functions: OpenStax §1: Functions and Graphs

Definition: Functions

A **function** consists of a set of inputs, outputs, and a rule for associating inputs to outputs. The valid inputs are the **domain** of the function, and the valid outputs are the **range**.

Example: Various Functions

- Inputs: Students. Outputs: Numbers. Rule: Every student has a student number.
- Inputs: Students. Outputs: Numbers. Rule: Every student has a height.
- Inputs: $\mathbb{R}$, Outputs: $\mathbb{R}$. Rule: To the number x assign x^2.

Question: A Human Range

Do you think that Height $= 10$ meters is in the range of students' heights?

Question: A Mathematical Domain

Is -1 in the domain of the function $f(x) = x^2$?

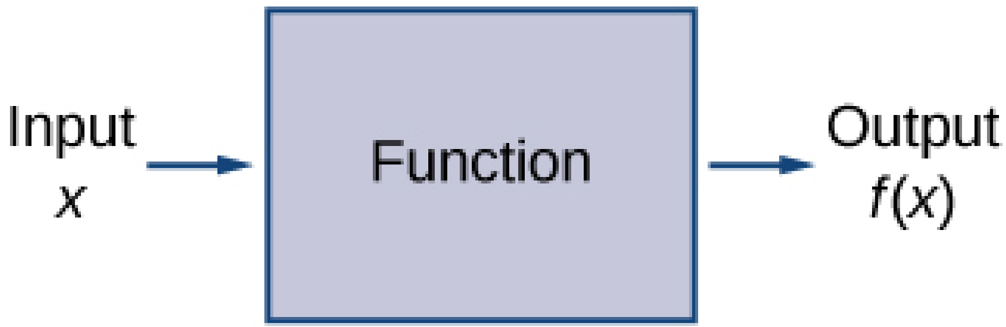
Question: A Mathematical Range

Is -1 in the range of the function $f(x) = x^2$?

Remark: Ways of Thinking of Things

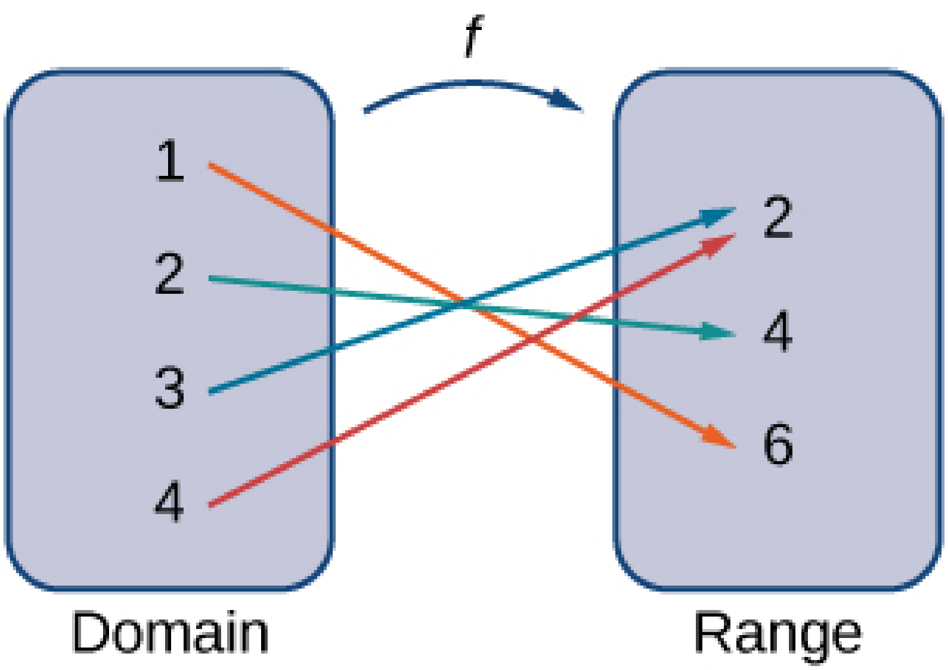
In mathematics, it is often helpful to have multiple ways of thinking about the same thing. Some ways of thinking about things work better for some people.

Functions as Black Boxes



OpenStax §1.1 Figure 1.2

Functions as Mappings

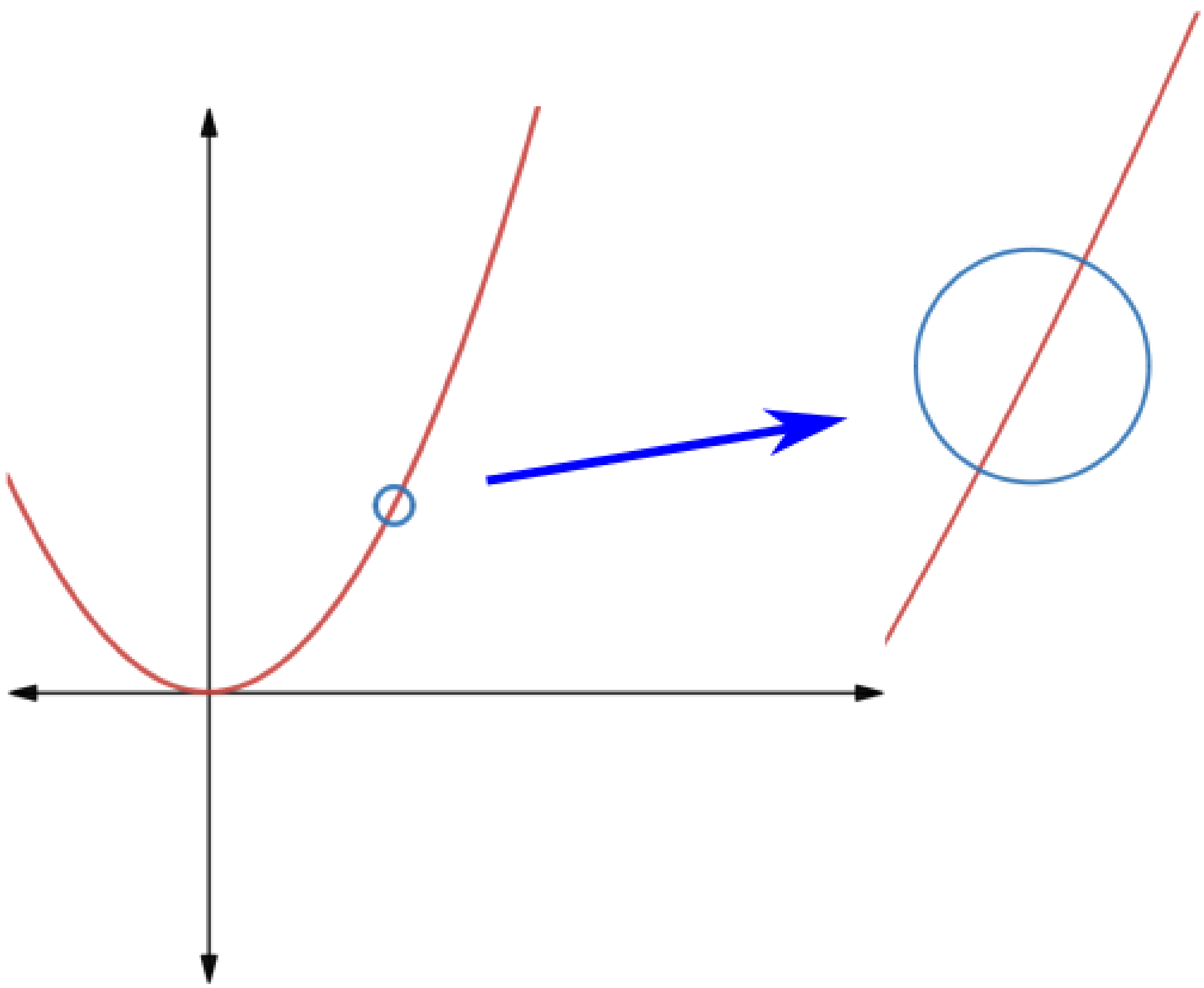


We can also present this data as a **table of values**:

x	1	2	3	4
$f(x)$				

Example: A Preview of Calculus

If we zoom in very close on any “nice” graph, we get a line.



Activity: Try This in Desmos

Open up Desmos, and type in a function. Zoom in on any point, and it’ll eventually become a line.

Question: “Nice” Functions

What do we mean by a nice function? What are the ingredients of such a function?

Definition: Lines: OpenStax §1.2: Basic Classes of Functions

A **linear function** is $y = mx + b$ where m and b are some numbers.
The **x-intercept** is where $y = 0$ and the **y-intercept** is where $x = 0$.

Question: Intercepts

Which of the following lines has x-intercept $x = 2$ and y-intercept $y = 4$?

1. $y = 2x + 4$
2. $y = 4x - 2$
3. $y = 4x + 2$
4. $y = -2x + 4$

Activity: Micro-Assignment: 5 min

Find an equation of the form $y = mx + b$ with
x-intercept $x = -3$ and y-intercept $y = 12$.

Definition: Lines: Slope **OS §1.2 Eq 1.3**

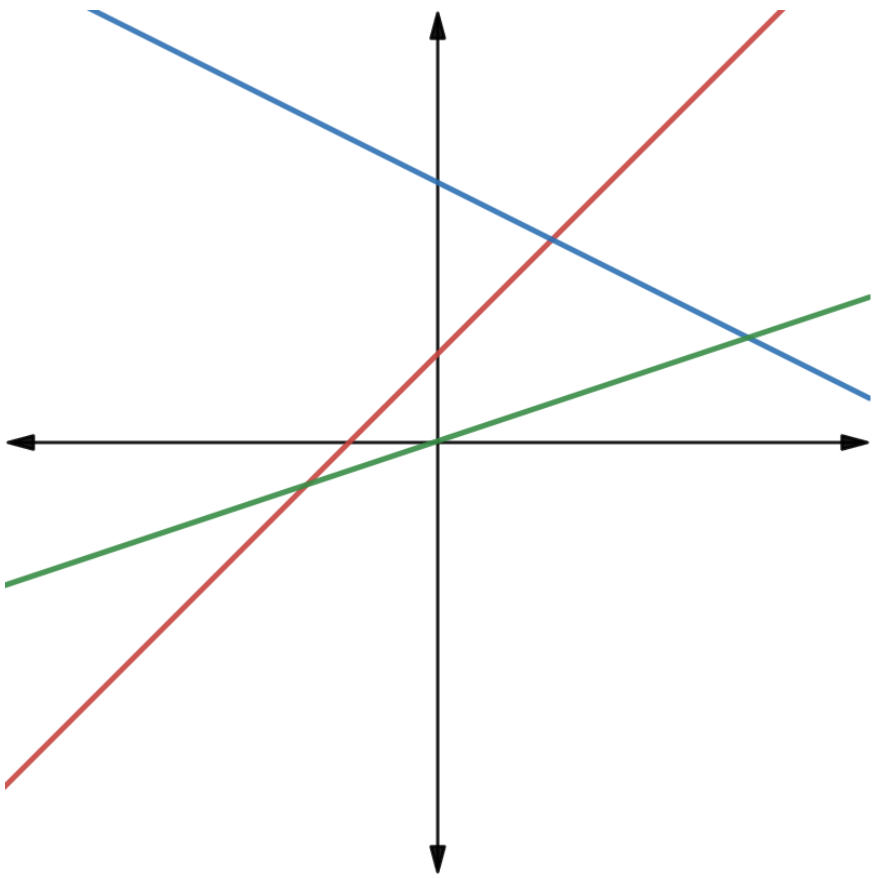
The **slope** of a line L through points (x_0, y_0) and (x_1, y_1) is:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$

where: $\Delta y = y_0 - y_1 =$ "rise" and $\Delta x = x_0 - x_1 =$ "run".

Activity: Discuss: 3 min

Which line has the greatest slope? Which line has the smallest y-intercept?
Sort the lines according to (i) their slope, and (ii) their y-intercept.
Which lines are increasing? Which are decreasing?



Question: Find A Line

Find the equation of a line $y = mx + b$ passing through the point $(2, 9)$ with slope $m = -3$.

Question: A Funny Domain And Range

What are the domain and range of the line $f(x) = 42$?

🏃 Activity: Class Discussion: 2 min

What are some other possible domains and ranges of lines?

Definition: Intervals — OpenStax §1.1

An **interval** is a collection of numbers defined inequalities: $\leq$, $\geq$, $<$, $>$.
For example,

- $\{x : 1 < x < 2\} = (1, 2)$
- $\{x : \pi \leq x < 4\} = [\pi, 4)$
- $\{x : x < 1\} = (-\infty, 1)$
- $\{x : -3 \leq x\} = [-3, \infty)$

We use [brackets for $\leq$ inequalities, and (brackets for $<$ inequalities.
Similarly, we use] brackets for $\geq$ inequalities, and) brackets for $>$ inequalities.

Definition: The Real Numbers

The <u>real numbers</u> are $(-\infty, \infty) = \mathbb{R}$.

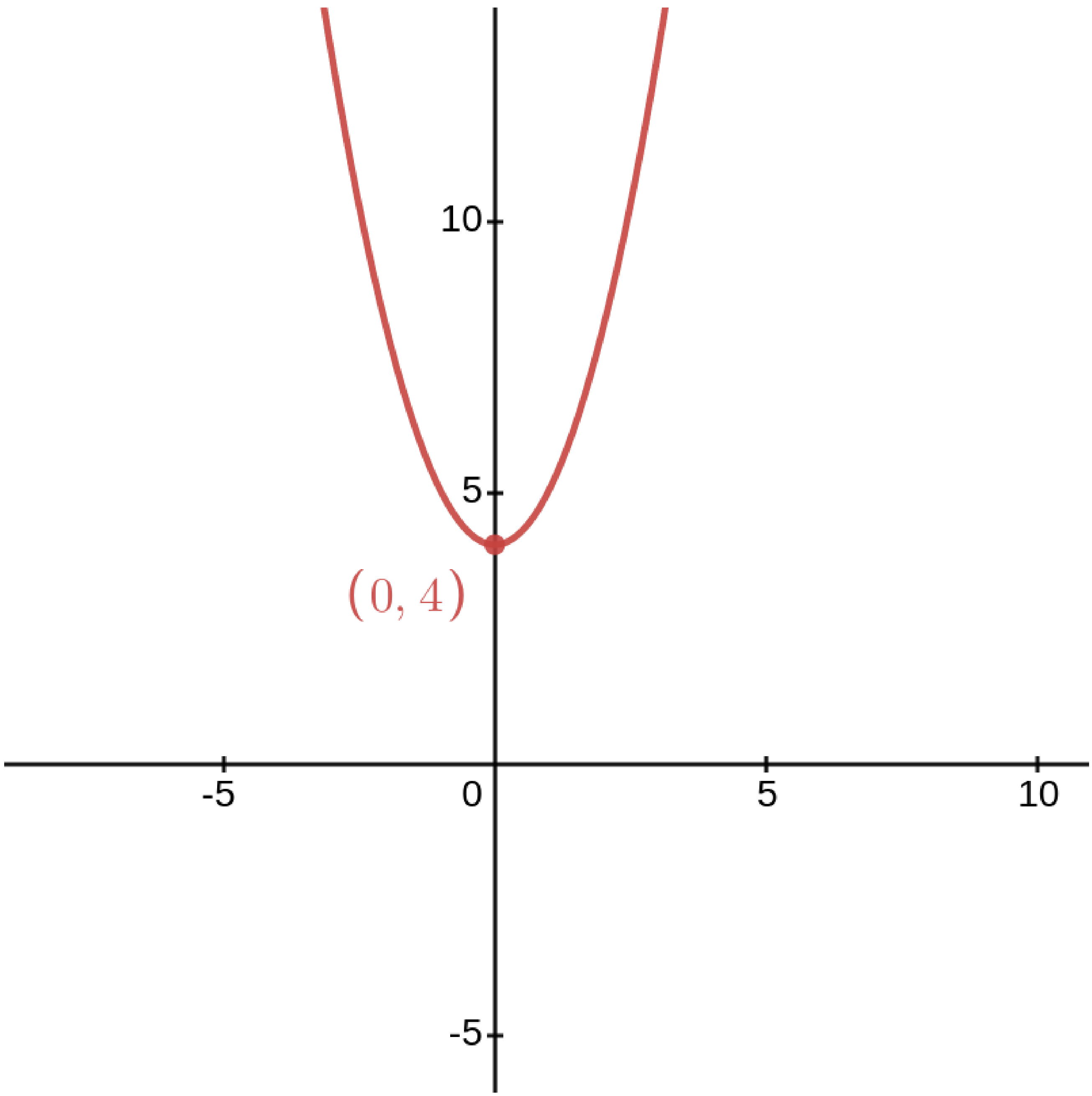
Question: Find the Interval

Which of the following intervals represents: $\{x : x \leq -2\}$?

1. $(-2, \infty)$
2. $(-\infty, -2)$
3. $(-\infty, -2]$
4. $[-2, \infty)$

Question: Find the Domain and Range

What are the domain and range of $f(x) = x^2 + 4$?



Remark: Writing matters.

In this class, I will heavily emphasize writing out full solutions. This is (probably) quite different from your experience in highschool where only the final answer is graded. We're doing this because written communication is a highly valuable skill. (Everyone wants good communication!)

Let's write out a full solution for: "Find the domain and range of $f(x) = x^2 + 4$."

- Domain:

- Range:

https://www.reddit.com/r/UTSC/comments/1fpdfpl/how_much_work_do_i_need_to_show_for_a_single_math/

Definition: Composition

Given two functions $f(x)$ and $g(x)$ one can **compose** them:

$$f(g(x)) = (f \circ g)(x) = \text{“f of g of x”} \qquad g(f(x)) = (g \circ f)(x)$$

This is operation is very important later in the course when we study the chain rule.

Question: Two Simple Compositions

Write out the composition $(f \circ g)(x)$ and $(g \circ f)(x)$ for $f(x) = \sqrt{x}$ and $g(x) = x + 9$.

Example: Composition and Transformation

How does the composition $g(x) = \sin(2x)$ modify the graph of $y = \sin(x)$?
Highlight any key points in both graphs.

Activity: Micro-Assignment: 5 min

What are the domain and range of $f(x) = \sqrt{x-6}$?

Question: Domain of a Rational Function

What is the **domain** of $f(x) = \frac{1}{x-3}$?

Question: Range of a Rational Function

What is the **range** of $f(x) = \frac{1}{x-3}$?

Definition: Piecewise Functions

A **piecewise** function has different equations for different parts of its domain.

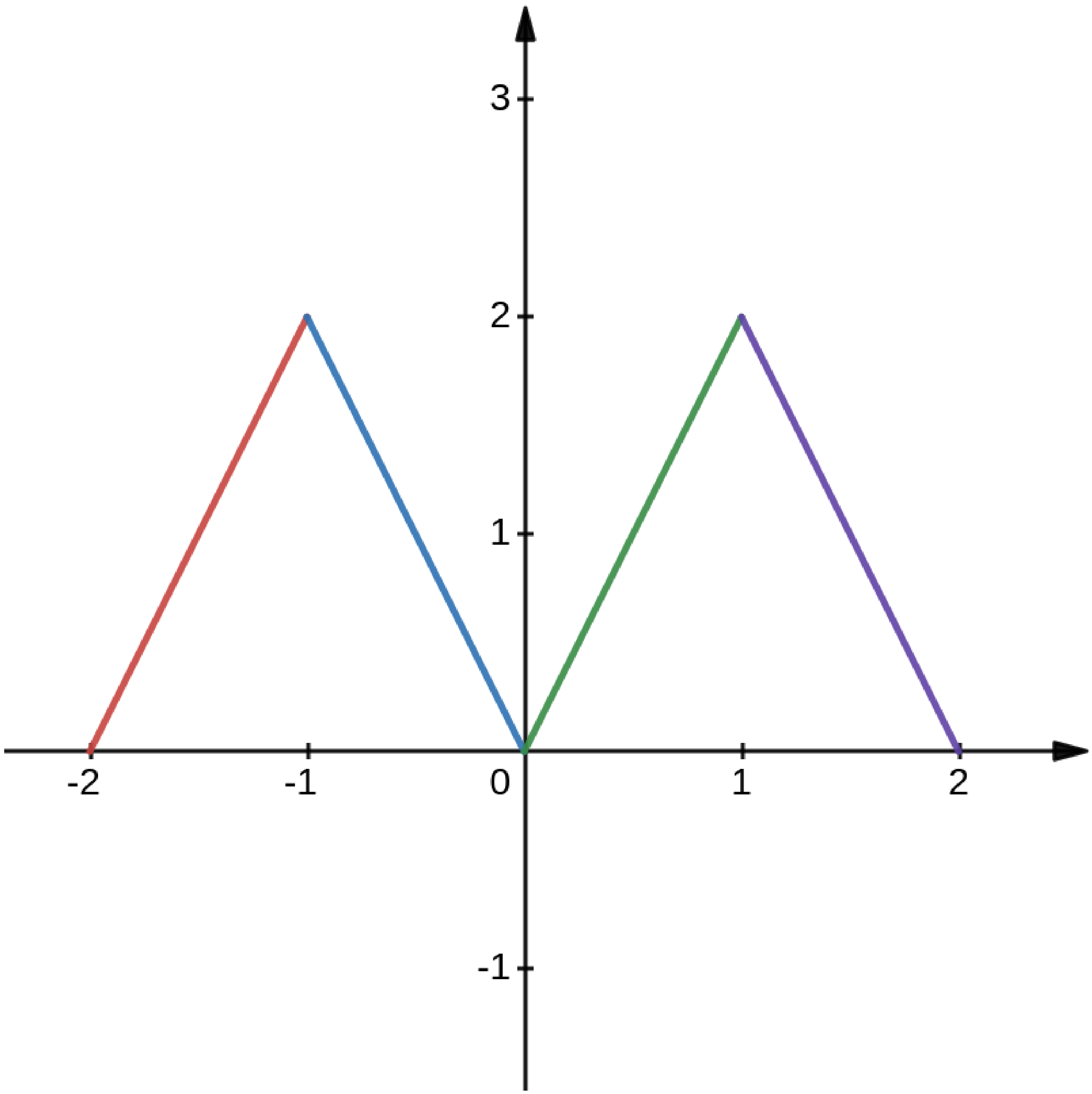
Question: Sketch A Function

Sketch the function:

$$f(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x^2 & x > 0 \end{cases}$$

🏃 Activity: Class Discussion: 3 min

What can you say about this piecewise graph using the material we learned this week?



Example: An Application: Femur Strength

A femur (or thighbone) is a bone in your leg. It is essentially a long thin tube filled with marrow. It has outer radius R and inner radius r where $R > r > 0$. The density of bone is approximately 1.8g/cm^3 and marrow is approximately 1g/cm^3. A medically significant ratio of a bone is: $k = R/r$. Suppose that a femur has length $L > 0$ in centimeters. (Adapted from Starr, Cecie, A. Evers Christine, and Starr Lisa. *Biology: concepts & applications.* Cengage Learning, 2018.)

1. Sketch a diagram of a femur (as a long thin tube) and label it with R, r, and L.
2. What are the possible values of k?
3. Express the mass $m(k)$ of the femur as a function of k.
4. What is the domain of $m(k)$? What is the range $m(k)$?
5. The following question is not a math question, it requires some thinking about biology:
   If an elderly person falls, do they want $k \approx 1$ or $k \approx 10$?

Week 2: Polynomials

Definition: Polynomials — OS §1.2 Eq. 1.7

A **polynomial** is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

The **degree** d of a polynomial is the highest power in it.
The number in front of the highest power is the **leading coefficient**.

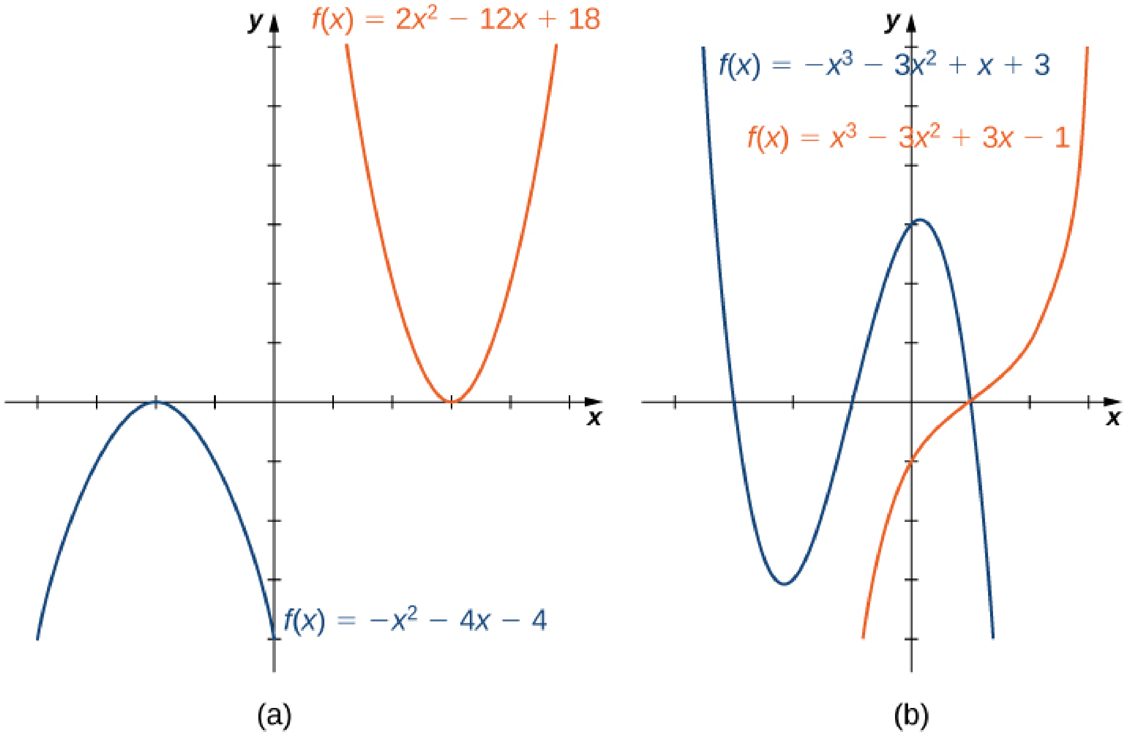
Example: Coefficients

Write out the coefficients of:

$$f(x) = 2x^3 + 9x^2 = 2x^3 + 9x^2 + 0x + 0.$$

Definition: End Behaviour of Polynomials

The end behaviour of $f(x)$ is what $f(x)$ does when x gets very large ($x \to \infty$).



OpenStax §1.2 Figure 1.19

Example: End Behaviour Visually

Describe the end behaviour of the functions above.

Question: Describe End Behaviour

Using the words "DEGREE d" and "LEADING COEFFICIENT a" make a short paragraph which describes the end behaviour of polynomials of even degree.

Activity: Micro-Assignment: 5 min

Use the words

- DEGREE d
- LEADING COEFFICIENT a
- THE $x \to \infty$ END
- THE $x \to -\infty$ END

to make a short paragraph explaining the end behaviour of polynomials of odd degree.

Definition: Symmetry

A graph has **even symmetry** if $f(x) = f(-x)$. It has **odd symmetry** if $f(x) = -f(-x)$.

Example: Even and Odd

The functions $f(x) = x^2,\ x^4,\ x^6$ have even symmetry.
The functions $f(x) = x,\ x^3,\ x^5$ have odd symmetry.

Activity: Discuss: 1 min

Why are these called even and odd symmetries?

Activity: Make Some Examples: 5 min

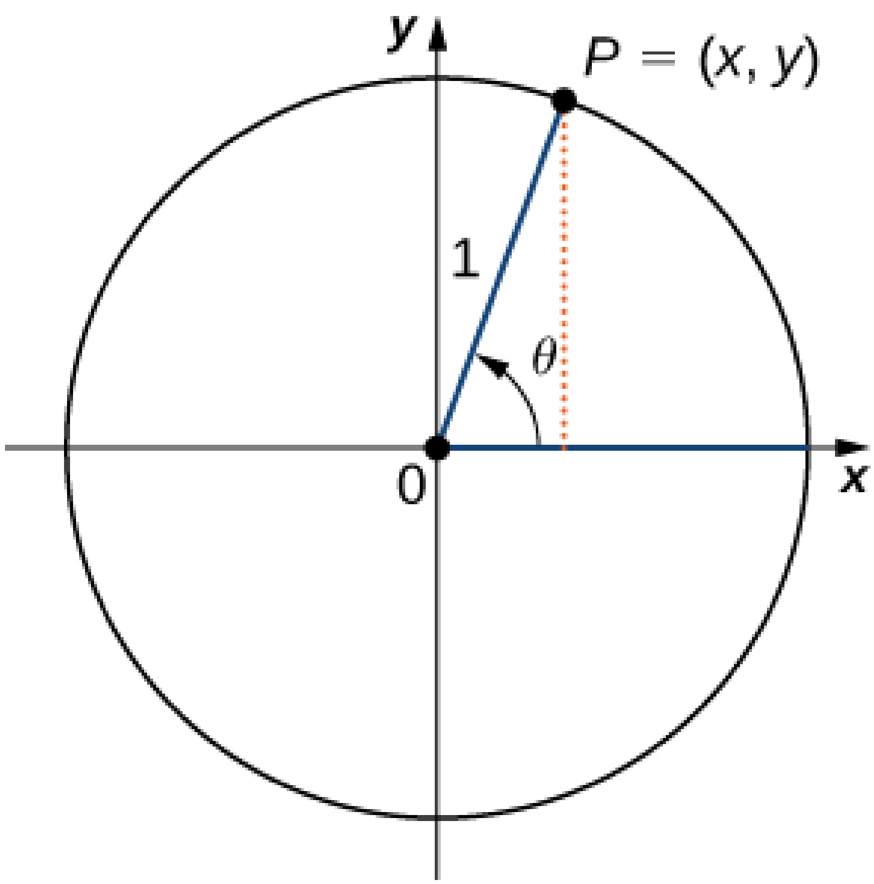
Whenever we meet two similar mathematical properties, it is important to investigate how they are related by finding examples. Find an example of $f(x)$ for each box.

		Even: Yes	Even: No
Odd	Yes		
	No		

Definition: The Trigonometric Functions

On the unit circle $x^2 + y^2 = 1$ we have:

$$(x, y) = (\cos(\theta), \sin(\theta)) = (\cos(-\theta), -\sin(-\theta))$$



OpenStax §1.3 Fig 1.31

Question: Working Backwards

Write the function $f(x) = \dfrac{1}{x^2+3}$ as a composition $(a \circ b \circ c)(x)$ for three functions: $a(x), b(x), c(x)$.

$$f(x) = \frac{1}{x^2+3} = \frac{1}{(((x)^2)+3)} = \frac{1}{(((x)^2)+3)}$$

Question: A Complicated Range

Find the range of $f(x) = \dfrac{1}{x^2+3}$.

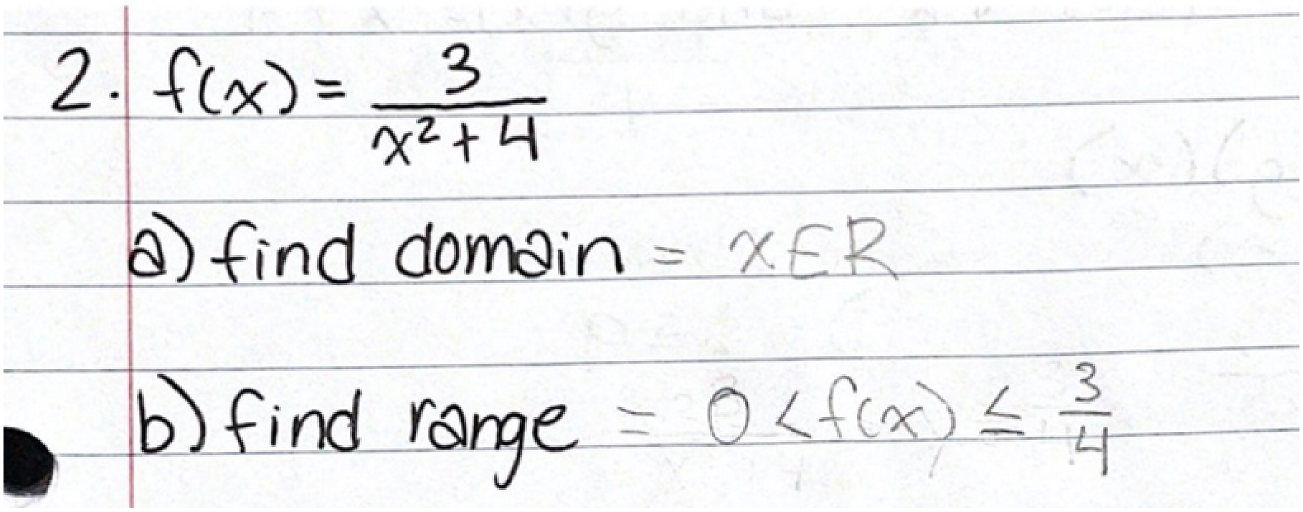
Graph it in Desmos. Notice that the graph is contained in $R = \left(0, \dfrac{1}{3}\right]$. How do we explain this?

Remark: An Important Observation

Look at the following pair of solutions to the problem.

$$\text{Find the domain of } f(x) = \frac{3}{x^2+4}.$$

What do you notice about the ratio of words to symbols?
Who do you think would score a higher mark?

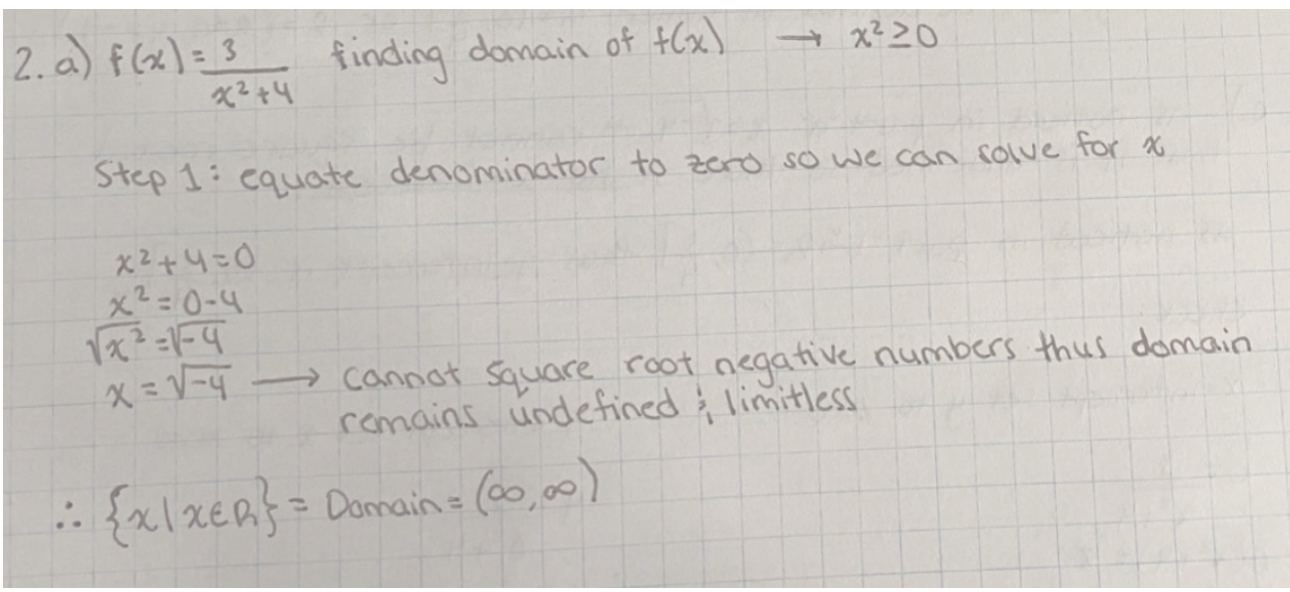


2. $f(x) = \dfrac{3}{x^2+4}$

a) find domain = $x \in R$

b) find range = $0 < f(x) \leq \frac{3}{4}$

Student Solution #1



2. a) $f(x) = \dfrac{3}{x^2+4}$ finding domain of $f(x)$ $\longrightarrow$ $x^2 \geq 0$

Step 1: equate denominator to zero so we can solve for x

$x^2+4=0$
$x^2 = 0-4$
$\sqrt{x^2} = \sqrt{-4}$
$x = \sqrt{-4}$ $\longrightarrow$ cannot square root negative numbers thus domain remains undefined ∴ limitless

∴ $\{x \mid x \in R\}$ = Domain = (∞, ∞)

Student Solution #2

Question: A Piecewise Range

Write the domain and range of the function.

$$f(x) = \begin{cases} 1 & x \neq \pi \\ 42 & x = \pi \end{cases}$$

Activity: Solo Work: 2 min

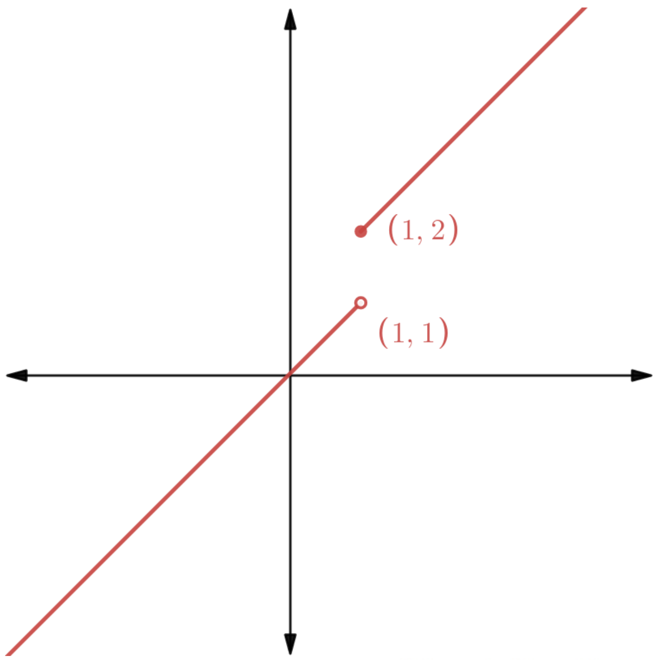
Sketch the function:

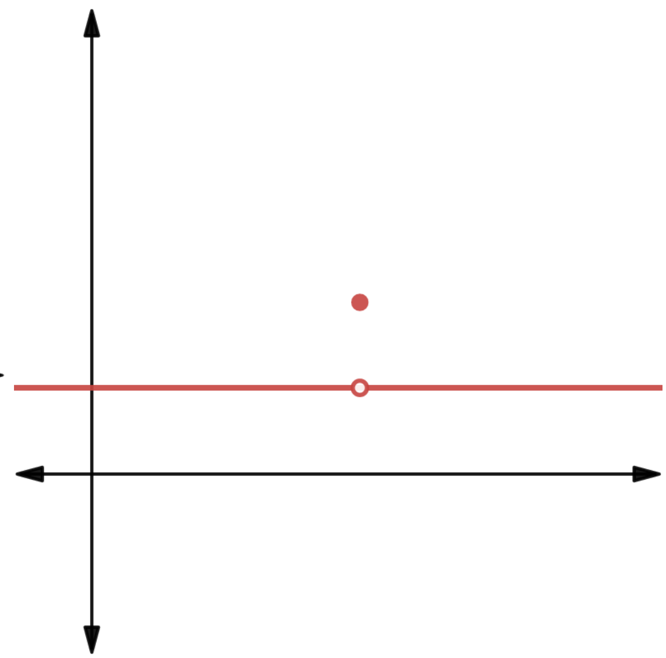
$$f(x) = \begin{cases} x & x < 1 \\ x+1 & x \geq 1 \end{cases}$$

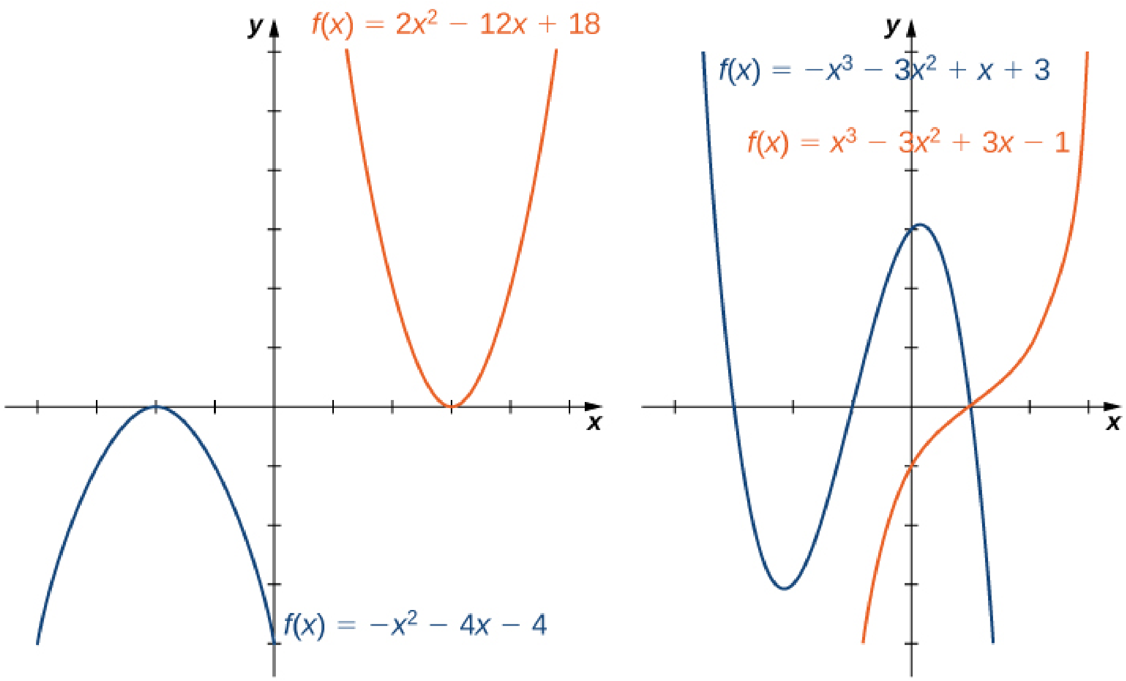
What do you notice about the values of $f(x)$ near $x = 1$?

Remark: Three Similar Problems

One of the "super powers" of mathematics is its ability to explain multiple phenomena using the same language. This is unifying process is called abstraction or generalization.







(a) (b)

Definition: Limits — OS §1.2 Defn 2.3

If $f(x)$ gets closer to L as x gets closer to a then $\lim_{x \to a} f(x) = L$.
We say that "the limit of $f(x)$ as x approaches a is L".

Question: A Strange Limit

What is the limit of $f(x)$ as x approaches π?

$$f(x) = \begin{cases} 1 & x \neq \pi \\ 42 & x = \pi \end{cases}$$

Question: Limits Visually

Compute the limit $\lim_{x \to 0} \frac{\sin(x)}{x}$ by inspecting the graph on Desmos.

Question: Limits Algebraically

Compute the limit $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

Definition: Two Sided Limits

Every number t has two sides:

the **left side** of t which we write t^- and the **right side** of t which we write t^+.

The **left hand limit** is $\lim\limits_{x \to t^-} f(x) = L$ if $f(x)$ approaches L when $x < t$.
The **right hand limit** is $\lim\limits_{x \to t^+} f(x) = L$ if $f(x)$ approaches L when $x > t$.

Question: A Jump

The function $f(x)$ has a "jump" from $(1, 1)$ to $(1, 2)$.

$$f(x) = \begin{cases} x & x < 1 \\ x + 1 & x \geq 1 \end{cases}$$

Describe the nature of this jump using limits.

Theorem: The Limit Laws

1. $\lim_{x \to a} c = c$
2. If n is a non-negative integer, then $\lim_{x \to a} x^n = a^n$.
3. $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
4. $\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
5. $\lim_{x \to a} f(x) \times g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$
6. If $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x)/ \lim_{x \to a} g(x)$.
7. If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ provided both limits exist.

Note: You will be provided these laws on the term test.

Example: Using the limit laws

Calculate and justify the following limit using the limit laws.

$$\lim_{x \to 1} \frac{x^2 - 3x + 7}{x + 1}$$

Note: This is long and boring, but we need to do it.

Definition: Does Not Exist

If $f(x)$ does not get close to any number L as x approaches a then the limit $\lim_{x \to a} f(x)$ <u>does not exist</u> (DNE). Alternatively, the limit is <u>not well defined</u>.

- *Why do we say "does not exist"?*
  We say this because there is no value L that the function gets close to.
  The value L is the thing which does not exist.

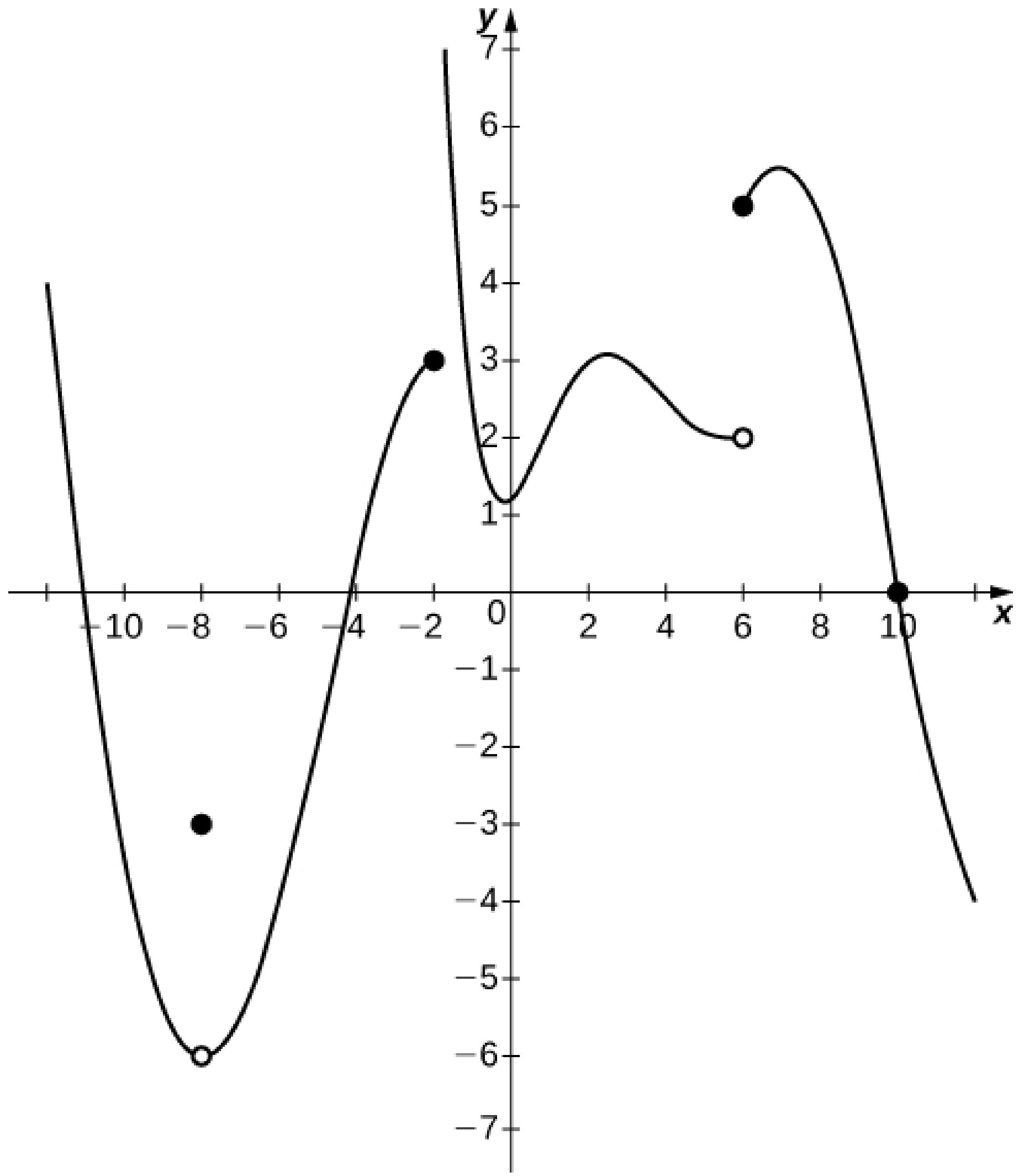
- *Why do we say "the limit is not well-defined"?*
  We say this because the notation $\lim_{x \to a} f(x)$ is ambiguous if:

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

- These two terms are equivalent.
  You probable heard "DNE" in highschool
  Parker might say "is not well-defined".

Question: A Bunch of Limits

Analyze the following graph using the notion of limits.



Activity: Check-In: 5 min

Please answer the following on a sheet of paper.

1. How is first year going?
2. How do you feel about this course?
3. What is something that you want to know?

Parker will walk around the room and chat with people.
Hand in your paper when you're done.

Summary of Week 2

- Polynomials: leading coefficients, degree
- Even and odd symmetry
- Composition
- Limits

Week 3: Limits and Derivatives

Definition: OS §1.2 Defn 2.3

If $f(x)$ gets closer to L as x gets closer to a then $\lim_{x \to a} f(x) = L$.
We say that "the limit of $f(x)$ as x approaches a is L".

Activity: A Limit Numerically

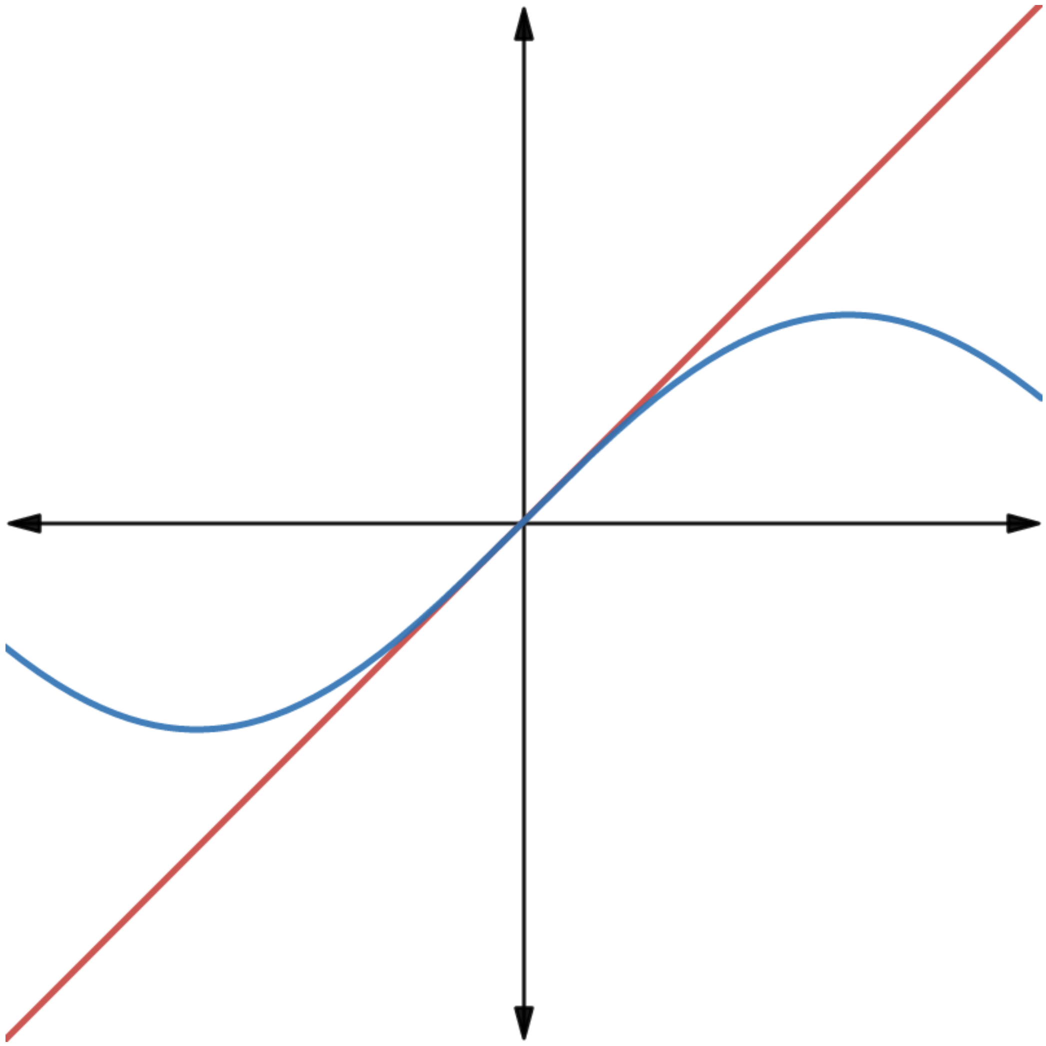
Set your calculator to radian mode.

Complete the following table of values of $f(x) = \dfrac{\sin(x)}{x}$ for $x = 0.1, 0.01, 0.001$.

x	0.1	0.01	0.001
$f(x)$			

Question: Investigate A Limit Graphically

Graph $y = \sin(x)$ and $y = x$ on the same pair of axes and zoom in on the point $(0, 0)$.
Write a paragraph describing what happens and how it related to $f(x) = \frac{\sin(x)}{x}$.



https://www.desmos.com/calculator/dfbwtcocof

Theorem: OpenStax Theorem 2.2

Suppose that L is a real number.

$$\lim_{x \to a} f(x) = L \Longleftrightarrow \left(\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L \right)$$

Comments:

- If both limits "go to the same number" then the limit is well-defined.
- Both sides of the limit need to match.

Example: A Life Sciences Example

Hemoglobin (Hb) is the iron-rich protein in blood cells that binds with oxygen (O2) in the lungs and exchanges it for carbon dioxide (CO2) in the tissues. In 1910, British physiologist Archibald Hill developed an empirical model to describe the binding of O_2 to Hb as[a]:

$$h(P) = \frac{P^k}{30^k + P^k} \qquad (k \geq 1)$$

where h is the proportion of hemoglobin molecules that are bound to O_2, P is the concentrations of O_2 measured as the partial pressure ($0 \leqslant P_{O_2} < \infty$), and k is the Hill coefficient.

1. Determine $\lim_{P \to 0} h(P)$ and interpret your results.
2. Does the limit in part (1) change for different values of k? Explain.
3. The half-saturation value $P_{50\%}$ is the concentration of oxygen at which the proportion of bound hemoglobin molecules reaches half its saturation value. Determine $P_{50\%}$.

[a] Hill AV. The possible effects of the aggregation of the molecules of haemoglobin on its dissociation curves. *J Physiol.* 1910; 40: iv–vii.

Question: An Infinite Limit

Use a table of values to support the following statement:

$$\text{“}\lim_{x \to 0} \frac{1}{x} \text{ does not exist.”}$$

Activity: Solo Work: 2 min

Use a table of values to support the following statement:

$$\text{“}\lim_{x \to 0} \frac{1}{x^2} \text{ is infinite.”}$$

Remark: Existence and Infinity: A Quote from Our Book

It is important to understand that when we write statements such as $\lim_{x\to a} f(x) = +\infty$ or $\lim_{x\to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x\to a} f(x) = +\infty$, we always write $\lim_{x\to a} f(x) = +\infty$ rather than $\lim_{x\to a} f(x)$ DNE. [**OpenStax §2.2 p.146**]

Activity: Discuss with Your Neighbour (3 min)

- What do you make of this quote?
- Do infinite limits exist? Why or why not?

Question: Compute a Limit: Plugging-In Doesn't Help

Evaluate the limit $\lim_{x\to 1} \frac{x^2 - 3x + 2}{x - 1}$.

For ease of reference, here are the limit laws.

Theorem: The Limit Laws

1. $\lim_{x \to a} c = c$
2. If n is a non-negative integer, then $\lim_{x \to a} x^n = a^n$.
3. $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
4. $\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
5. $\lim_{x \to a} f(x) \times g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$
6. If $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x)/ \lim_{x \to a} g(x)$.
7. If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ provided both limits exist.

Note: You will be provided these laws on the term test.

Question: Compute a Limit Using Known Data

Use the limit laws to evaluate

$$\lim_{x \to 1} (x^2 - x + 1) f(x)$$

if you know $\lim_{x \to 1} f(x) = 3$.

Definition: Slope (OpenStax §1.2 Eq 1.3)

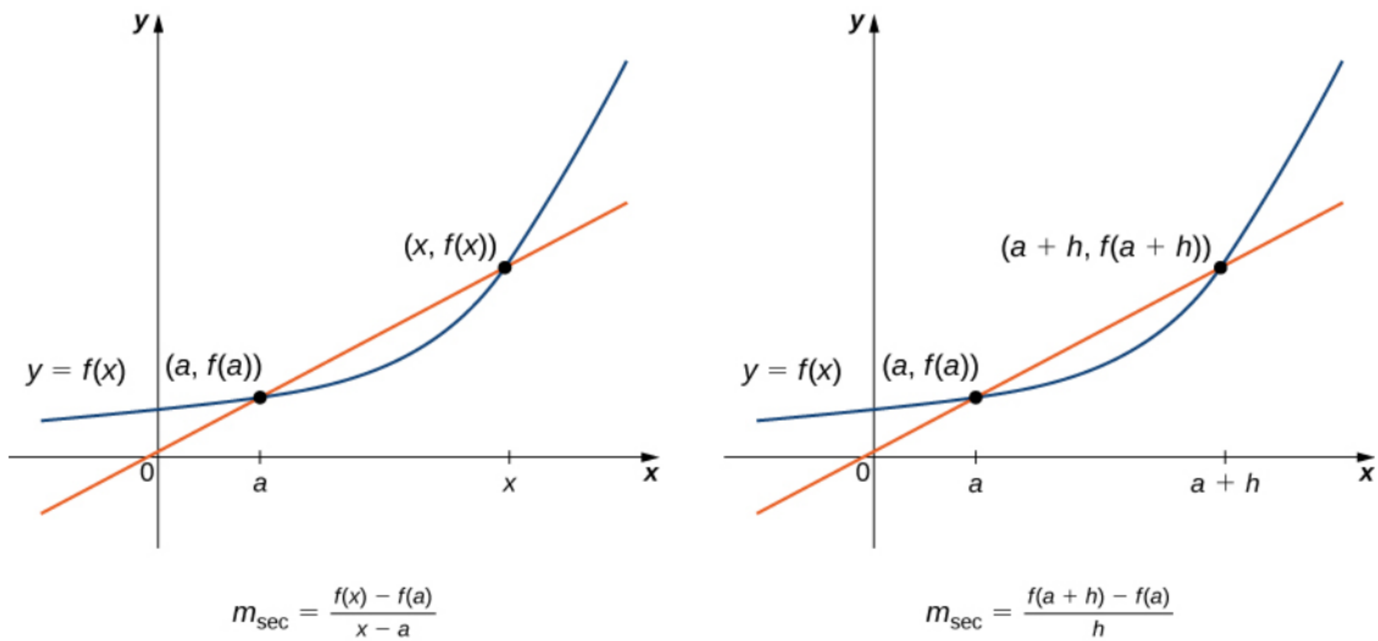
The **slope** of a line L through points (x_0, y_0) and (x_1, y_1) is:

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = m$$

Definition: Secant Slope

The **secant line** between two points $(a, f(a))$ and $(x, f(x))$ is a line segment joining those points. The **secant slope** is:

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$



OpenStax §3.1 Fig 3.3

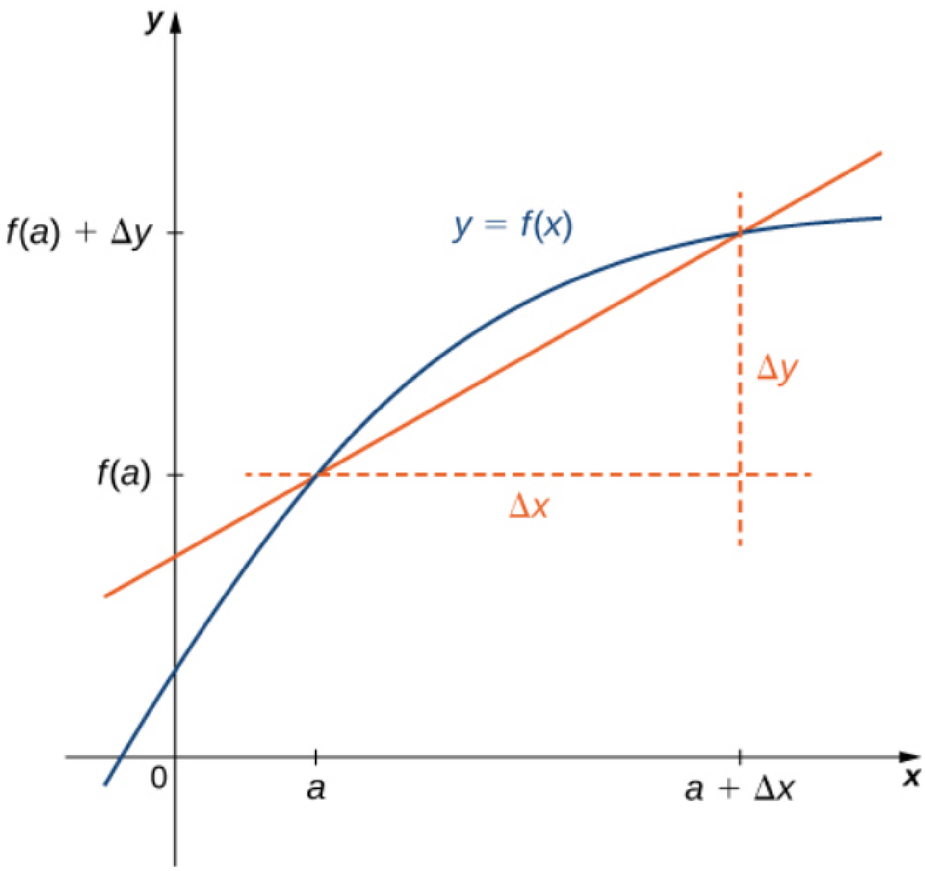
Definition: Derivatives: OpenStax Pg. 220

We define the **derivative** of $f(x)$ at $x = a$ to be:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \text{“ } f \text{ prime at } a \text{ ”}$$

We will also use the following **Leibniz notation**:

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{“ dee } y \text{ dee } x \text{ ”}$$



$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

OpenStax Pg. 234

Remark: Do we have to learn both notations?

Yes, absolutely and without a doubt. You need to be comfortable writing and interpreting both notations.

“I will learn Leibniz notation.”

Question: A Derivative from the Definition

Find the slope of $f(x) = x^2 + x$ at $a = 1$ using the definition of the derivative.

Remark: Frequently Asked Questions

Do we always have to do it this way?

No – We're doing things this way for now, so that I can teach you what a derivative really means. This is a really awkward way to compute a derivative.

Must we always show this much work?

Sort of – You should always show your work, and explain each step.

Can we use derivative rules?

Yes – Once we learn them.

Example: Working Backwards to a Derivative

Consider the following limit:

$$\lim_{h \to 0} \frac{e^h - 1}{h}$$

Which function $f(x)$ is this the derivative of? At what point a? What is the value of $f'(a)$?
Silly Joke: This limit is really Canadian, e^h?

Activity: Micro-Assignment (5 min)

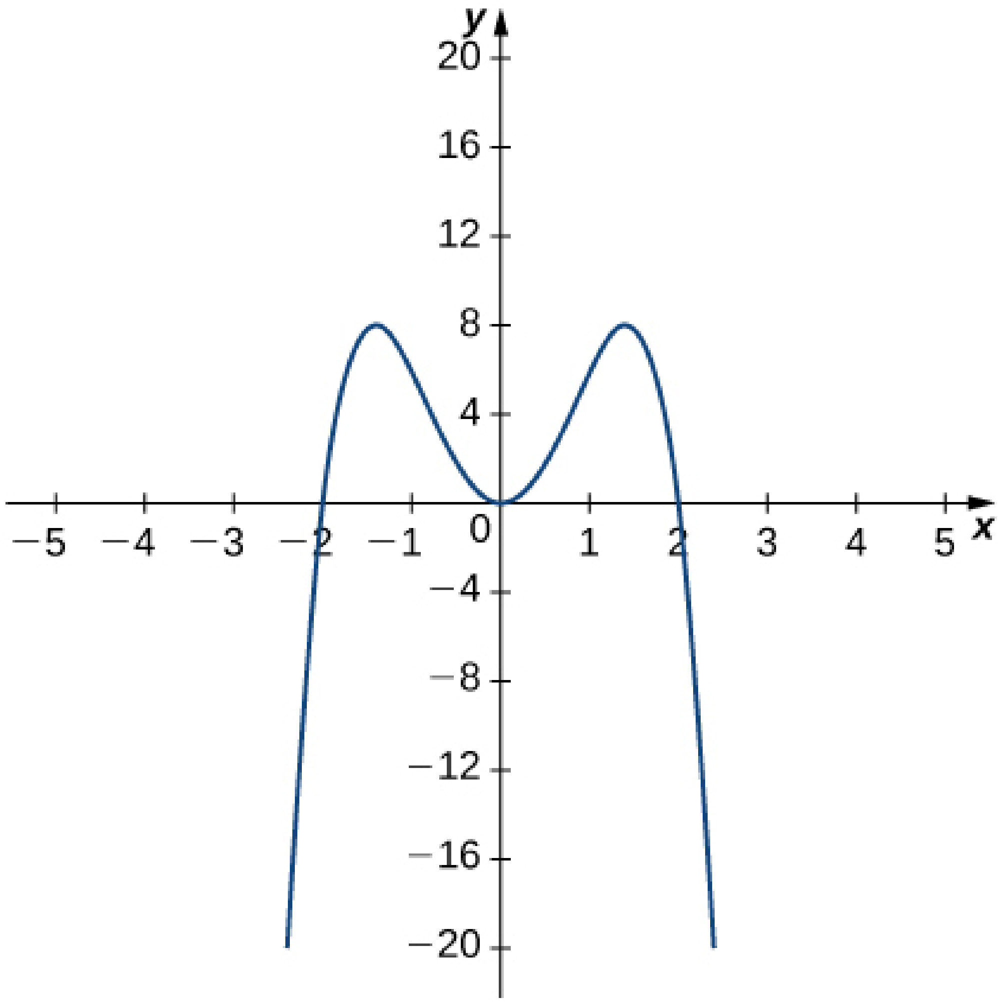
Find the slope of $f(x) = \frac{1}{x}$ using the definition of the derivative where $a \neq 0$.

Definition: The Derivative Function

The **derivative** of $f(x)$ is the function $f'(x)$. It measures the slope of $y = f(x)$ at each point $(x, f(x))$.

Example: Sketching a Slope Visually

Use the following graph of $y = f(x)$ to sketch the derivative function $y = f'(x)$.



OpenStax §3.2 Q65.

Definition: Tangent Lines

The **tangent line** to $y = f(x)$ at $x = a$ passes through $(a, f(a))$ and has slope:

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example: A Tangent Line to Square Root

Find the tangent line to $y = \sqrt{x}$ at $a = 4$.

Summary of Week 3

- Limits: from the left, from the right
- Existence of limits
- Infinite limits
- Derivative and slope

Week 7: Derivatives and The Shape of Graphs

Definition: The Mean Value Theorem

A function is **differentiable** on (a, b) if its derivative is defined for all $c \in (a, b)$.

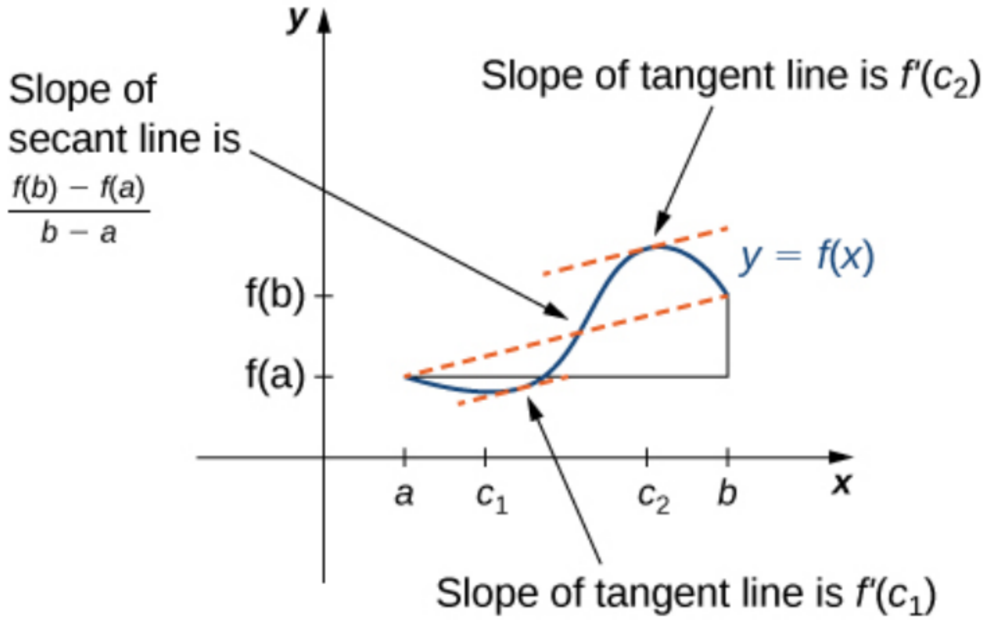
Theorem: OpenStax p.382

If $f(x)$ is differentiable on $[a, b]$ then there exists a point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Activity: Think-Pair-Share (3 min)

How can we re-phrase this theorem in simpler terms? Without a formula? Think about things like slopes and secants.



OpenStax Fig. 4.25

Example: A Strange Example

What does the Mean Value Theorem (MVT) say about $f(x) = |x|$ on $(-2, 2)$?

Activity: Class Poll: Differentiability

What is the derivative of $f(x) = x|x|$ at $x = 0$?

1. -2
2. 0
3. 2
4. f is not differentiable at $x = 0$

Theorem: Derivatives Detect Constants

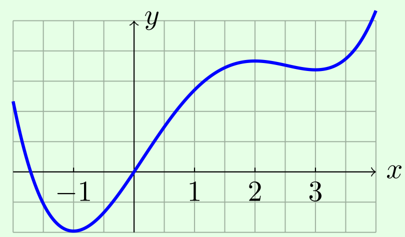
If $f'(x) = 0$ for all x then $f(x)$ is constant and does not change.

Theorem: Rolle's Theorem

Suppose f is differentiable. If $a < b$ and $f(a) = f(b)$ then $f'(c) = 0$ for some $c \in (a, b)$.

Activity: Class Discussion (2 min)

Consider the differentiable function f whose graph is given below.
Where is $f'(x) = 0$?



1. $x = 0, \frac{5}{2}$
2. $x = -1, 2, 3$
3. $x = -\frac{5}{2}, 0$
4. There are no points at which $f'(x) = 0$.

Follow-Up: When is $f'(x) > 0$ and when is $f'(x) < 0$? What do you notice?

Definition: Increasing and Decreasing

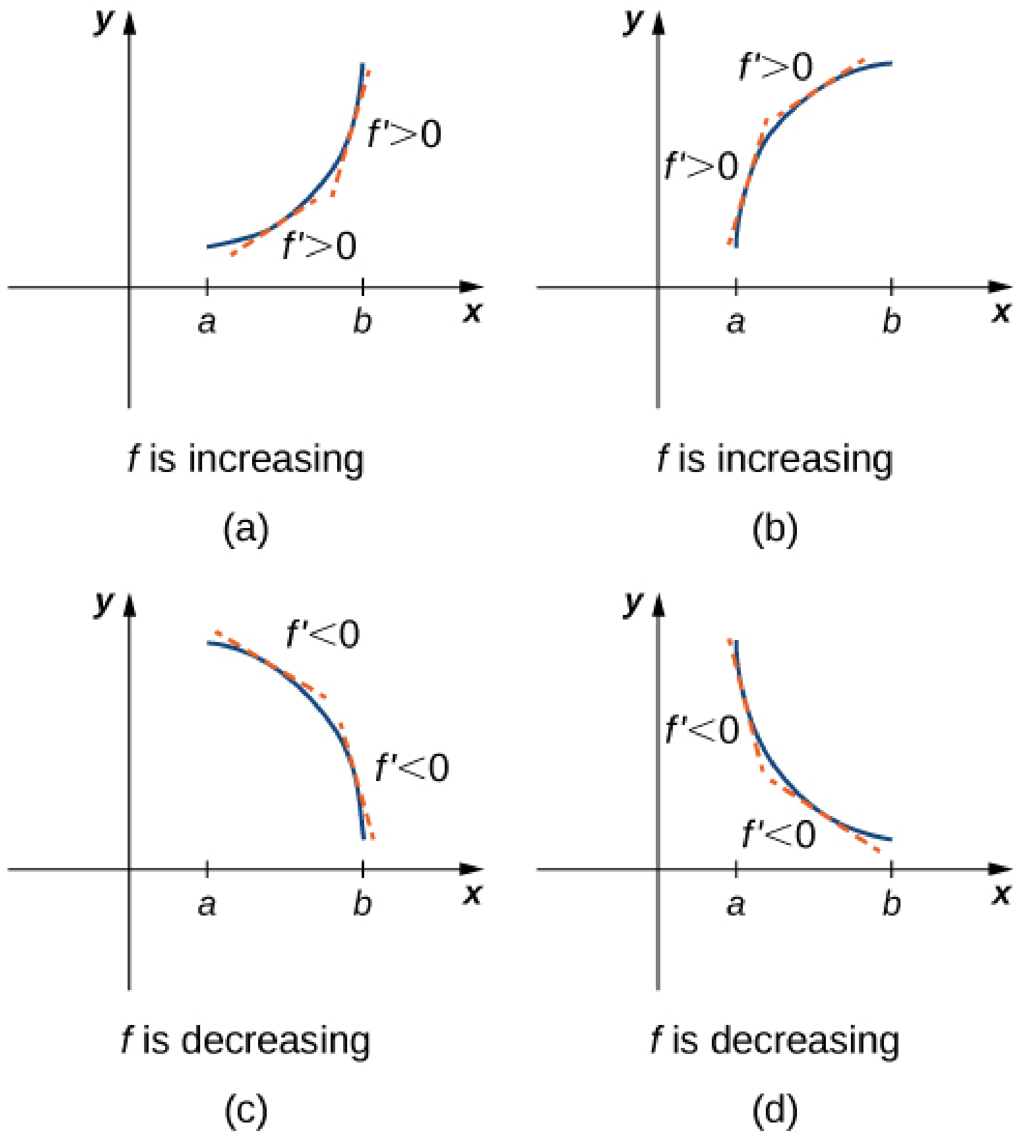
If $a < b$ implies $f(a) < f(b)$ then $f(x)$ is **increasing**. ("Bigger input implies bigger output.")
If $a < b$ implies $f(a) > f(b)$ then $f(x)$ is **decreasing**. ("Bigger input implies smaller output.")

Theorem: Derivatives Detect Increasing

If $f'(c) > 0$ for all values of c then $f(x)$ is increasing.

Activity: Think-Pair-Share (3 min)

What is the corresponding statement which describes when f is decreasing?



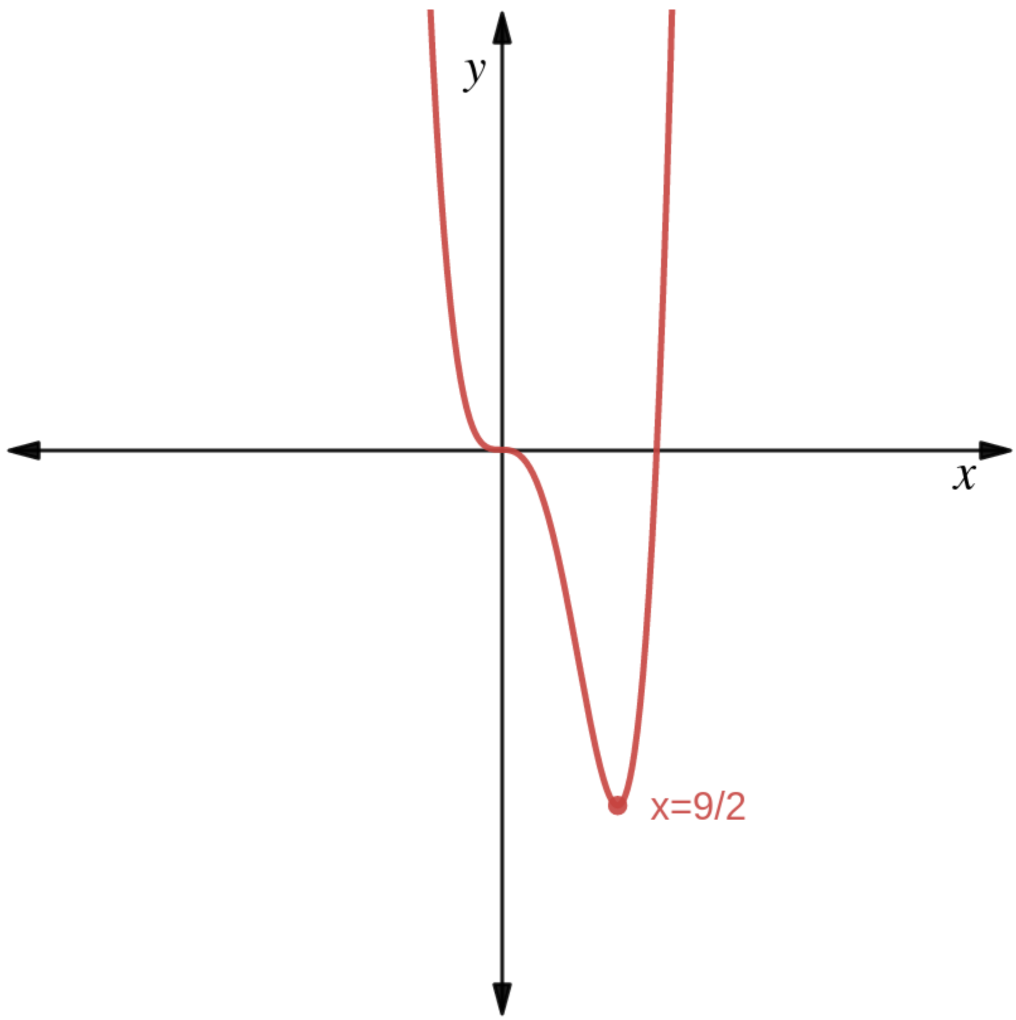
OpenStax Figure 4.30

Question: When Is A Function Increasing? OpenStax §4.3 Q226]

When is $f(x) = x^4 - 6x^3$ increasing?

Example: Local Minimum

If $f(x) = x^4 - 6x^3$ is increasing when $x > \frac{9}{2}$ and decreasing when $x < \frac{9}{2}$, then what can we conclude about $x = \frac{9}{2}$? What about $x = 0$?



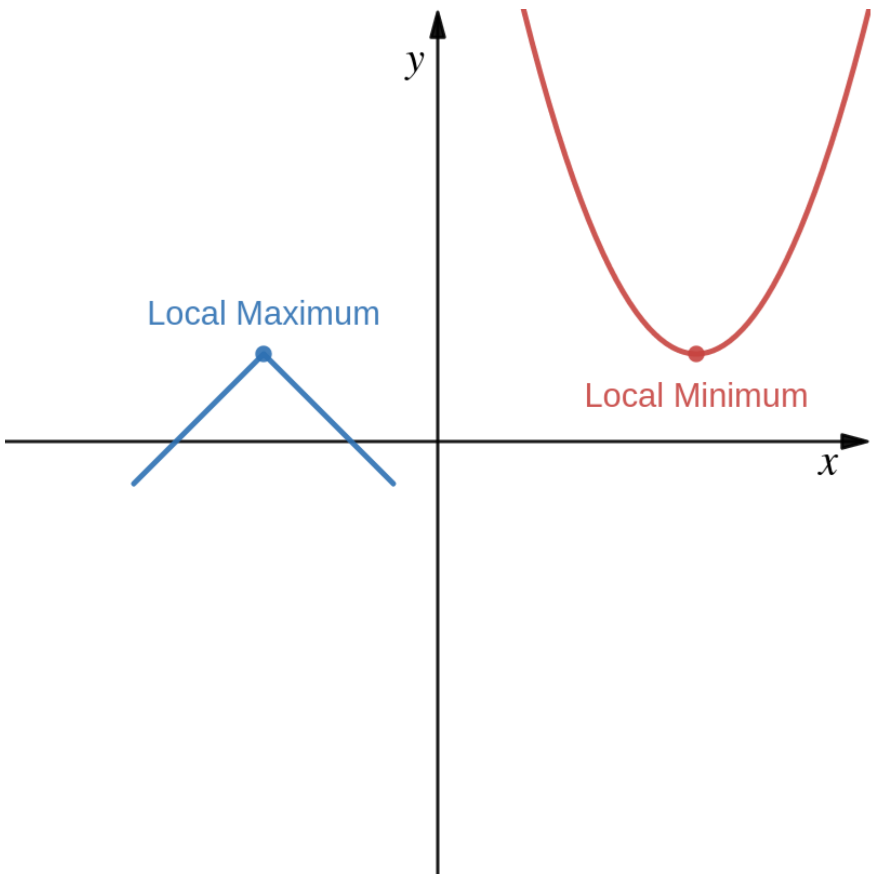
Theorem: The First Derivative Test **OpenStax p.392**

Suppose $x = c$ is a critical point of $f(x)$.

If $f'(x)$ changes sign from $(+)$ when $x < c$ to $(-)$ when $x > c$ then:
$x = c$ is a local maximum.

If $f'(x)$ changes sign from $(-)$ when $x < c$ to $(+)$ when $x > c$ then:
$x = c$ is a local minimum.

If there is no change of sign, then $x = c$ is neither a max nor a min.

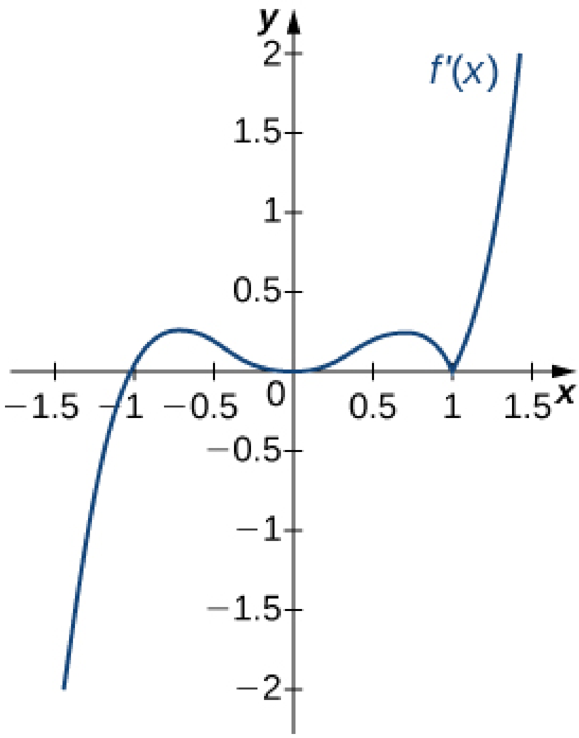


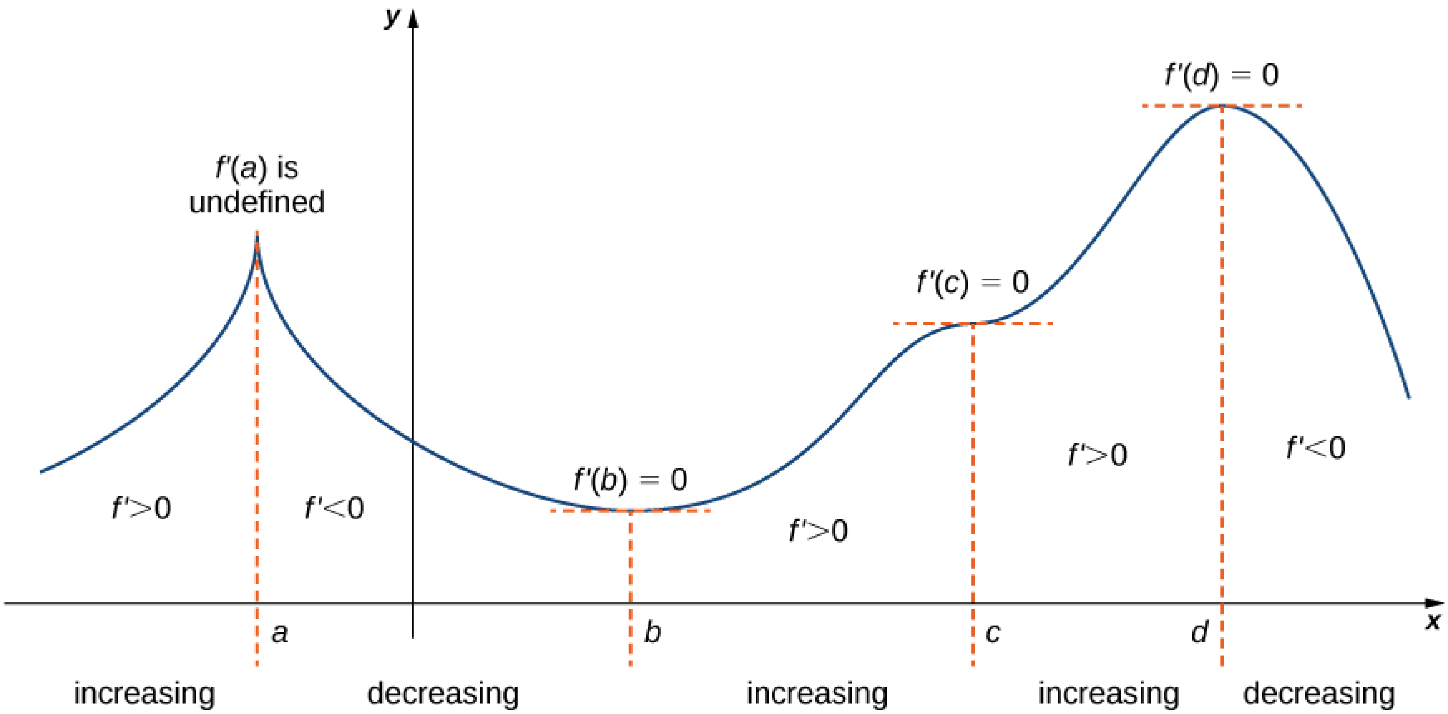
Example: Using the First Derivative Graph

Consider the graph below. Where is $f(x)$ decreasing?

1. $(-0.75, 0) \cup (0.75, 1)$
2. $(1, \infty)$
3. $(-\infty, -1)$
4. $(0, 1)$

Follow-up question: What else can you determine about $f(x)$ from this graph?





OpenStax Figure 4.31

Question: Building a Sign Chart

Suppose that $f'(x) = 2x^2(2x-9)$. Find the locations of all local extrema.

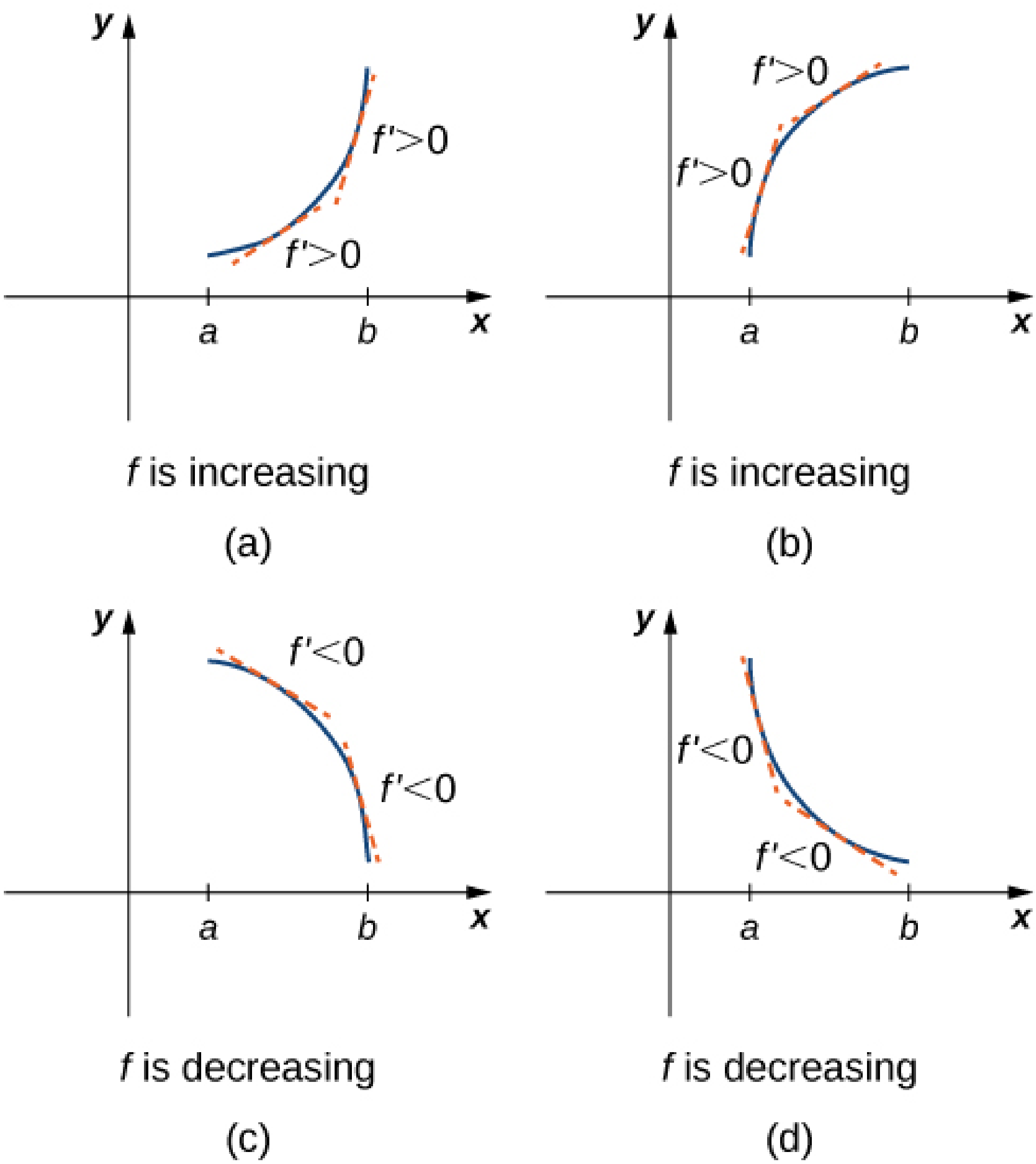
Interval	**Test Point**	**Sign of $f'(x)$**	**Conclusion**
$(-\infty, 0)$	$x=-1$		
$(0, 9/2)$	$x=1$		
$(9/2, \infty)$	$x=10$		

Problem-Solving Process:

- Create intervals using the critical points $x=c$.
- Pick a "test point" in each interval.
- Evaluate the derivative at the test point.
- Conclude whether $f(x)$ is increasing / decreasing on each interval.
- Form a chart of this information.

Activity: Increasing and Decreasing: What do you notice?

What do you notice about the graphs below?



OpenStax Figure 4.30
We need some new vocabulary!

Definition: Higher Derivatives

If $f(x)$ is a function then $f'(x)$ is its **first derivative**.
The **second derivative** of $f(x)$ is:

$$f''(x) = \frac{d}{dx}\left[f'(x)\right] = \text{“the derivative of the first derivative”}$$

We can also write:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

In general, the n'th derivative is:

$$f^{(n)}(x) \qquad \frac{d^ny}{dx^n}$$

Definition: Concavity OpenStax p. 395

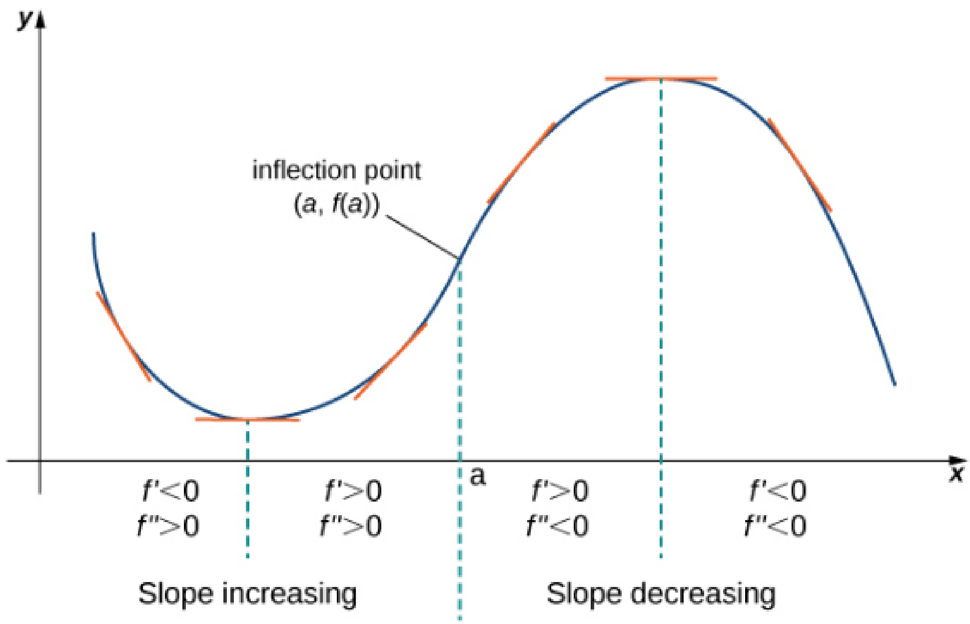
If f' is increasing, then we say that f is **concave up**.
If f' is decreasing, then we say that f is **concave down**.

Theorem: Second Derivatives Detect Concavity

If $f'' > 0$ then f is concave up. If $f'' < 0$ then f is concave down.
Warning: $f''(a) = 0$ does not imply a change of concavity.
$y = x^4$ has $\frac{d^2y}{dx^2} = 4 \cdot 3 \cdot x^2 = 0$ at $x = 0$ but $y = x^4$ is always concave up.

Definition: Points of Inflection

If f changes concavity at $x = a$ then $(a, f(a))$ is a **point of inflection**.



OpenStax Fig 4.35

Theorem: The Second Derivative Test **OpenStax p. 400**

Suppose $f'(c) = 0$ and $f''(x)$ is continuous.

1. If $f''(c) > 0$ then $x = c$ is:
2. If $f''(c) < 0$ then $x = c$ is:
3. If $f''(c) = 0$ then the test is inconclusive.

Example: Using The Second Derivative Test

Find and classify the extrema of $f(x) = 3x^5 - 5x^3$.

Example: A Curve Sketch!

Sketch the graph of $f(x) = x^3 - 3x$.

- Find the x and y intercepts.
- Find the critical points of $f(x)$.
- Make a sign chart for $f'(x)$.
- Classify the critical points.

Summary of Week 7

- Increasing and decreasing
- Concave up and down
- Second derivative test
- Sign charts

Week 8: Curve Sketching, Differentials, and Approximation

Remark: Mish-Mash Warning

This week has a bit of everything. We're going to do a bunch of small things.

Question: Sketching from Given Data

Draw a graph $f(x)$ defined on the interval $[-7, 7]$ with all of the following properties:

1. A local maximum at $f(5) = 5$
2. A local minimum at $f(-5) = -5$
3. A critical point that is not a maximum or minimum at $x = 0$.

Definition: Rational Functions — OpenStax p. 47

A **rational function** is a function $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.
Rational functions are helpful because they:

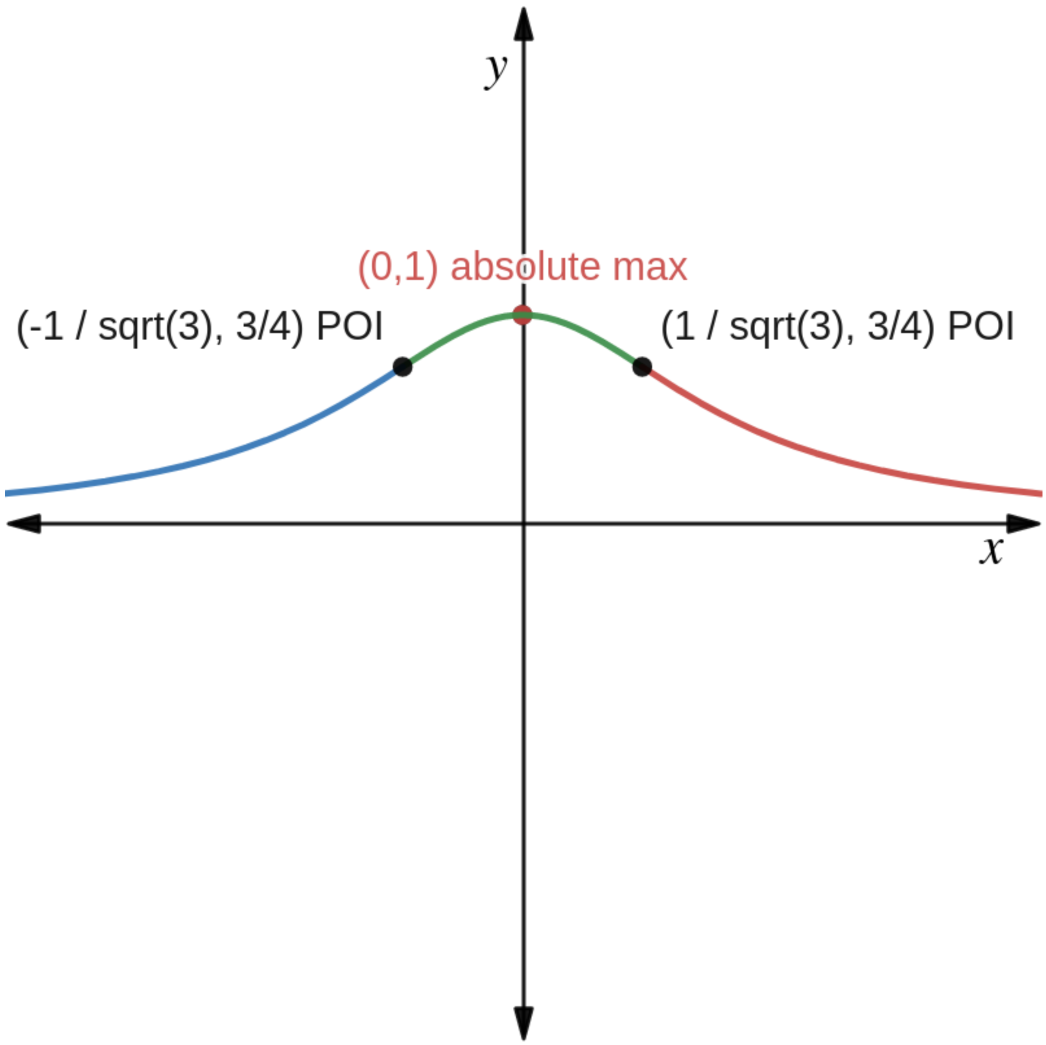
- Often have vertical asymptotes.
- Easy to analyze end-behaviour.
- Complicated curve sketches.

Example: A Rational Curve Sketch

Sketch the graph of $f(x) = \frac{1}{1+x^2}$.

- Find the x and y intercepts.
- Find the critical points of $f(x)$.
- Make a sign chart for $f'(x)$.
- Classify the critical points.
- Locate points of inflection.
- Analyze asympotes and end-behaviour.

(This page continues the example: Sketch the graph of $f(x) = \frac{1}{1+x^2}$.)



https://www.desmos.com/calculator/brz4ngiuiv

Definition: Vertical Asymptotes — OpenStax p. 149

Consider a function $f(x)$. We say $x = a$ is a **vertical asymptote** if any of the following occur:

1. $\lim_{x\to a^+} f(x) = \pm\infty$
2. $\lim_{x\to a^-} f(x) = \pm\infty$
3. $\lim_{x\to a} f(x) = \pm\infty$

Definition: Horizontal Asymptotes — OpenStax p. 408 (!)

A function $f(x)$ has a **horizontal asymptote** $y = L$ either of the following occur:

1. $\lim_{x\to\infty} f(x) = L$
2. $\lim_{x\to-\infty} f(x) = L$

Example: Find Vertical Asymptotes

Determine where $y = \dfrac{x-1}{x^2-3x+2}$ has vertical asumptotes.

Example: Find Horizontal Asymptotes

Determine the horizontal asymptotes of $y = \dfrac{x-1}{x^2-3x+2}$.

🏃 Activity: Asymptotic Trickiness (5 min)

Find an example or justify each of the following claims about asymptotes:

- A function can have many vertical asymptotes.
- A function can only have two horizontal asymptotes.
- A function can cross its horizontal asymptotes.
- A function cannot cross its vertical asymptotes.

Sketch your examples using Desmos.

Example: A Curve Sketch with Asymptotes

Sketch the curve $y = \dfrac{1}{1-x^2}$ assuming:

$$\frac{dy}{dx} = \frac{2x}{(1-x^2)^2} \qquad \frac{d^2y}{dx^2} = \frac{6x^2+2}{(1-x^2)^3}$$

Remark: Log-Log and Semi-Log Plots

We introduce the notions of log-log graphs and semi-log graphs. These are (probably) the first new topics in the course for some students. Unfortunately, they're in the course description AND the textbook doesn't have any material on them.

Definition: Log-Log and Semi-Log Plots

To graph a **semi-log** graph:

- Calculate a list of points (x, y) on the graph.
- Plot each point: draw a point (x, y) which is x units from the y-axis and $\log(y)$ units from the x-axis.

To graph a **log-log** graph:

- Calculate a list of points (x, y) on the graph.
- Plot each point: draw a point (x, y) which is $\log(x)$ units from the y-axis and $\log(y)$ units from the x-axis.

Activity: Class Discussion (3 min)

The instructions above describe how to plot log-log and semi-log graphs. Formulate a similar set of instructions for plotting a standard cartesian graph like in highschool.

Example: Make a Semi-Log Plot

Consider the function $y = 10^x$.

1. Calculate five points where $x = 1, 2, 3, 4, 5$ on this graph.
2. Plot the five points from part (1) on a standard cartesian graph.
3. Plot the five points from part (1) on a semi-log graph.

Example: Make a Log-Log Plot

Consider the graph $y = x^2$.

1. Calculate five points where $x = 1, 2, 3, 4, 5$ on this graph.
2. Plot the five points from part (1) on a standard cartesian graph.
3. Plot the five points from part (1) on a log-log graph.

Theorem: The Power of Log Graphs

If $f(x)$ is an exponential function of the form $f(x) = Ae^{Bx}$ then the semi-log graph of $y = f(x)$ appears to be a straight line. If $g(x)$ is a power function of the form $g(x) = Ax^B$ then the log-log graph of $y = g(x)$ appears to be a straight line. *Big Idea*: Log graphs help us analyze data.

Example: Kepler's Third Law

Make a log-log plot of the following points (P, A) for each planet.

Planet	**Orbital Period (P)**	**Axis (A)**
Mercury	0.241	0.39
Venus	0.615	0.72
Earth	1.00	1.00
Mars	1.88	1.52
Jupiter	11.8	5.2
Saturn	29.5	9.54
Uranus	84.0	19.18
Neptune	165	30.06
Pluto	248	39.44

https://www.desmos.com/calculator/b54bankl0q

Example: Deduce a Power Law from Given Data

Use the log-log plot to find a relation of the form $A \approx P^c$ between A and P.
History: Kepler worked on this problem for decades. It is a major accomplishment.

Let us take two points from the above log-log graph,*i.e.*, Earth and Neptune.

For Earth, $(\log_{10}(P), \log_{10}(A)) = (0, 0)$.
For Neptune, $(\log_{10}(P), \log_{10}(A)) = (1.924, 1.282)$.

We know that slope of log-log graph is the exponent of power law relationship. Therefore, the slope of log-log graph is,

$$m = \frac{1.282 - 0}{1.924 - 0} = \frac{1.282}{1.924} = 0.666 \approx \frac{2}{3}.$$

Hence, the relationship between P and A is:

$$A \approx P^{2/3}.$$

Remark: Linear Approximations

Now we're going to go on a quick detour through the land of linear approximations. This is *really* just taking a tangent line. Nothing new here. This detour is a refresher to provide context for differentials.

Definition: Linear Approximations OpenStax p. 355

The **linear approximation** to $f(x)$ at $x = a$ is:

$$L(x) - f(a) = f'(a)(x - a)$$

Notice:

- This is the point-slope format of a line.
- The line has slope $m_{tan} = f'(a)$.
- The line passes through $(a, f(a))$.

Activity: Try It Out (2 min)

Find the tangent line of $f(x) = \sqrt{x}$ at $a = 4$.

Example: Approximate Using A Tangent Line

Use the tangent line $y = \frac{1}{4}x + 1$ of $y = \sqrt{x}$ at $a = 4$ to approximate $\sqrt{4.01}$.

$$\frac{1}{4}(4.01) + 1 = 2.0025 \qquad \sqrt{4.01} = 2.00249...$$

Definition: Differentials — OpenStax p. 359

The equation $\frac{dy}{dx} = f'(x)$ can be re-written as:

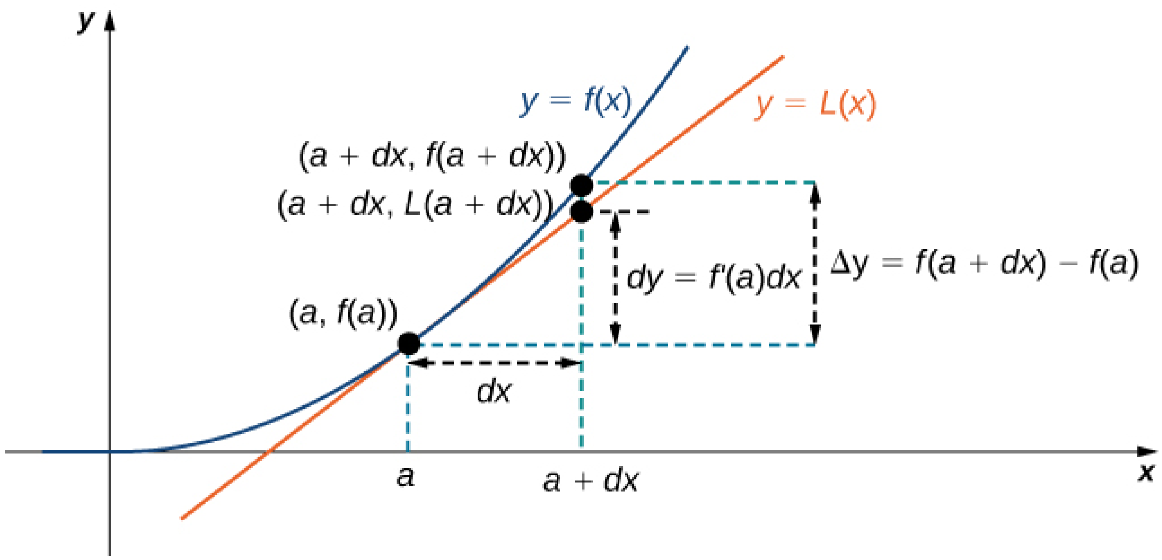
$$dy = f'(x)dx$$

The terms dy and dx are **differentials**.

Recall that:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The differentials dy and dx represent "small changes in y and x".



OpenStax Fig 4.11: The differential $dy = f'(a)dx$ is used to approximate Δy if: x increases from $x = a$ to $x = a + dx$.

Example: Find A Differential

Calculate Δy and dy when $y = x^2 + 2x$, $x = 3$, and $dx = 0.1$

Definition: Inverse Functions OpenStax p. 78

Suppose f has range R and domain D. If $f(x)$ and $f^{-1}(x)$ satisfy:

1. $f(f^{-1}(y)) = y$ for all $y \in R$
2. $f^{-1}(f(x)) = x$ for all $x \in D$

then f and f^{-1} are **inverse functions**.
Notation Warning: We can also write $f^{-1}(x)$ although this mixes up domain and range.

Example: Verify An Inverse

Check that $f(x) = \frac{1}{2}x + 1$ and $f^{-1}(y) = 2y - 2$ are inverse functions.

Activity: Find An Inverse

Find the inverse of $y = f(x) = x^2 + 1$ on the domain $[0, \infty)$.

- Solve the equation $y = x^2 + 1$ for x.
- Interchange x and y.
- Pick the correct range for $f^{-1}(x)$.

Week 9: Riemann Sums

Remark: A Lot of Machinery

This week we begin our study of the area bounded by curves. The main tool we'll develop is the theory of Riemann sums. This material is a *lot* more technical than the course has been so far. If it is overwhelming remember: people can learn this stuff. You can learn it to. You can always ask for help.

The plan is to calculate an area in a "machinery free" way and then build up the theory progressively.

Example: A Highschool Problem and A Hard Problem

Highschool: Calculate the area of the triangle bounded by the lines $y = 0$, $y = x$, and $x = 1$.
Hard: Find the area bounded by $y = 0$, $y = x^2$, and $x = 1$. (Archimedes is famous for solving this.)

Remark: Approximation!

The whole idea of Riemann sums rests on the idea of approximation. We take better and better approximations, until we get the actual area.

Example: Approximating The Triangle By Rectangles

The triangle T bounded by the lines $y = 0$, $y = x$, and $x = 1$ has base $[0, 1]$. Approximate the area of T by splitting the base in to two parts of equal length and erecting rectangles on bases. Write the **left end-point** approximation T_L and the **right end-point** approximation T_R separately.

Activity: Try It Yourself (5 min)

Repeat the previous example but split the base $[0, 1]$ of the triangle T in to three parts of equal length. Calculate T_L and T_R as before. What do you notice about the values T_L and T_R?

Remark: Why we need sequences.

Our approximations clearly depend on the number of pieces which we use to split up the triangle. We want a compact way to describe "the behaviour of T_L and T_R with n parts".

Definition: Sequences

A **sequence** x_n is a list of real numbers with a value for each n in the naturals.
We also write $x_n = x(n)$ as a function of n. We call x_n a **term** of the sequence, and n is the **index** of x_n.

Example: Some Common Sequences

Compute the first five terms $n = 1, 2, 3, 4, 5$ of the following sequences:

1. $x_n = n$
2. $x_n = 2^n$
3. $x_n = \dfrac{1}{n}$

Example: General Formulas for T_L and T_R

Suppose that we split the interval $[0, 1]$ in to n parts of equal length.
Write general formulas for $T_L(n)$ and $T_R(n)$.

Remark: Why we need summation.

As we can see, our formulas for $T_L(n)$ and $T_R(n)$ involve a lot of "dot dot dots". We want a compact way to describe these summations so that we can use algebra and other tools to handle sequences.

Definition: Summation

Given a sequence x_n we can define its **sequence of partial sums** by:

$$S_N = x_1 + x_2 + \cdots + x_N = \sum_{k=1}^{N} x_k.$$

The compact way of writing this involves **sigma notation**:

$$\sum_{k=a}^{b} x_n$$

We call k the **dummy variable** or **index of summation**. The values $k = a$ and $k = b$ are the **lower** and **upper bound** respectively. *Note*: We may start the summation at $k = 1$ or another other value. Other common choices of dummy variable are i and n.

Example: Calculate Some Partial Sums

Calculate the following sums:

1. $\displaystyle\sum_{k=0}^{2}(2k+1)$
2. $\displaystyle\sum_{k=0}^{2}\frac{1}{2^n}$
3. $\displaystyle\sum_{k=3}^{6}(-1)^k$

Example: Find a Formula

Evaluate the first four terms $N = 1, 2, 3, 4$ of the following and guess formulas for S_N:

1. $S_N = \displaystyle\sum_{k=0}^{N} \pi$
2. $S_N = \displaystyle\sum_{k=1}^{N} 42$

Example: Arithmetic Progressions

An **arithmetic progression** is $x_n = A + nB$. Find a formula for the partial sum $S_N = \sum_{k=0}^{N} x_k$.

Example: Geometric Series

A **geometric series** is a series of the form:

$$S_N = ar^0 + ar^1 + \cdots + ar^N = \sum_{k=0}^{N} ar^k$$

Show that $S_N = a\frac{1 - r^{N+1}}{1 - r}$. (*Story*: The case $a = 1$ and $r = \frac{1}{2}$ has a silly story about pouring two beers.)

Example: Little Gauss's Sum

Find a formula for the following:

$$S_N = \sum_{k=1}^{N} k.$$

Story: There is a famous story about the mathematician Gauss. When he was a little child, his teacher asked his whole class to add up the numbers from one to a hundred. In this notation, that question is "Calculate S_{100}." Gauss, the prodigy, instantly responded: 5050.

Example: The Riemann Sum Area of T

Setup a general formula for $T_L(n)$ and $T_R(n)$. Take the limit as n goes to infinity.

Note: We ought to get $A = 1/2$ by highschool geometry.

Activity: Class Discussion (5 min)

Look over our calculation of the area of T. Here are some questions to consider:

- What's the difference between $T_L(n)$ and $T_R(n)$?
- What were the basic ingredients of the calculation?
- What really depended on the function $y = x$?

Definition: Riemann Sums

The example of T has led us to develop the theory of Riemann sums. To be concrete, a **Riemann sum** is:

$$\text{"The signed area bounded by } y = f(x) \text{ on } [a,b]\text{"} = \lim_{N \to \infty} \sum_{k=0}^{N} f(x_k^*) \Delta x_k$$

This definition has a lot of sub-parts. We name them now:

- The **end-points** are a sequence x_k such that: $a = x_0 < x_1 < \cdots < x_N = b$.
- Δx_k is the **length** of the interval $[x_k, x_{k+1}]$.
- x_k^* is a **sample point** in the interval $[x_k, x_{k+1}]$.

https://pgadey.ca/notes/advice-for-students/#know-the-definitions

Week 10: The Fundamental Theorem of Calculus and Anti-Derivatives

Remark: Riemann Sums Are Hard

Last week, we introduced the machinery of Riemann sums. They are hard to calculate because we need a special formula for each sum. This week, we introduce a fundamental tool for calculating them which simplifies the process of computing areas.

Theorem: The Fundamental Theorem of Calculus (Version I)

If $f(x)$ is a nice function on $[a, b]$ then:

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Note: Our textbook calls this "Theorem 5.6: The Net Change Theorem".

Example: Calculate An Area (The Hard Problem)

Find the area bounded by $y = x^2$, $y = 0$, and $x = 0$ using the Fundamental Theorem of Calculus.

Example: Find An Area **OS §5.4 Q212**

$$\int_0^{\pi/2} x - \sin(x)\ dx$$

Remark: Integration Requires Us to Undo Differentiation

To apply the Fundamental Theorem of Calculus to a given formula:

$$\int_a^b g(x)\ dx$$

we need to write $g(x) = \dfrac{dF}{dx}$ for some $F(x)$. This means that we need to *undo* the process of differentiation.

Definition: Antiderivatives

We say that $F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$.
Note: What does $F(x)$ do? It is a function with slope $g(x)$.

Example: Multiple Antiderivatives

Check that $F_1(x) = \sin(x)$ and $F_2(x) = \sin(x) + 10$ are both antiderivatives of $f(x) = \cos(x)$.

Example: Checking Antiderivatives

Which of the following is an antiderivative of $f(x) = e^{x/2}$? Circle all correct answers.

$$e^{x/2} \qquad 2e^{x/2} \qquad 2e^{(1+x)/2} \qquad 2e^{x/2} + 1 \qquad e^{x^2/4}$$

Definition: Integral Notation

We write: $\int f(x)dx = F(x) + C$ to express "$F(x)$ is an antiderivative of $f(x)$". The term $+C$ is called the **constant of integration**. The term $\int f(x)dx$ is called an **indefinite integral**. If have definite bounds, we get a **definite integral**:

$$\int_a^b f(x) = F(b) - F(a)$$

Theorem: Algebra of Indefinite Integrals

We have the following relations:

$$\int kf(x)dx = k\int f(x)dx$$
$$\int f_1(x) + f_2(x)dx = \int f_1(x)dx + \int f_2(x)dx$$
$$\int f_1(x) - f_2(x)dx = \int f_1(x)dx - \int f_2(x)dx$$

Example: Powers

If $n \neq -1$ then we have: $\displaystyle\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$ (We'll handle $n = -1$ soon.)

Example: A Polynomial Example

Calculate $\displaystyle\int_0^2 x^2 + 3x + 1 \, dx.$

Remark: Every Derivative Rule Is Also An Antiderivative Rule

We can convert any derivative rule in to an anti-derivative by reversing the sides of the equality.

$$\frac{d}{dx}[F(x)] = f(x) \iff \int f(x)\ dx = F(x) + C$$

Activity: Class Discussion (5 min)

Produce a table of "anti-derivative rules". (Think about stuff like trigonometry, exponentials, etc.)

$f(x)$	$\int f(x)\ dx$

Theorem: Substitution

The chain rule corresponds to the following antiderivative rule.

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(g(x)) + C$$

Note: This process is often called u**-substitution**.

Example: Applying Substitution

Adjust the integrand as necessary to apply the substitution $u = 4x^2 + 9$ and calculate the indefinite integral.

$$\int \frac{x}{\sqrt{4x^2+9}}\,dx$$

Example: Substitution and Unmatched Terms

$$\int x(1-x)^{99}\ dx$$

Example: Substitution and Bounds

Evaluate the following definite integral. Change the bounds appropriately when applying substitution.

$$\int_0^1 x\sqrt{2-x^2}\,dx$$

Remark: Reversing The Product Rule

So far, we've seen that every derivative rule has a corresponding antiderivative rule. The chain rule becomes substitution. Now, we begin to reverse the sense of the product rule. Notice what this lets us go: We can exchange an integral $f'g$ for fg'. This lets us "move the difficulty of the integral around".

Theorem: Integration by Parts — OpenStax Vol II §3.1

The product rule gives us the following:

$$\frac{d}{dx}[fg] = f'g + fg' \Longleftrightarrow \int f'g + fg' \; dx = fg + C$$

We rarely have such a nice integrand, and so we re-arrange to get:

$$\int f'g \; dx = fg - \int fg' \; dx + C$$

Example: A Polynomial Times An Exponential

Calculate the following antiderivative: $\displaystyle\int_0^1 xe^x \; dx$.

Example: Which Parts?

Which parts should we choose for the following: $\int x^3 \ln(x)\ dx$?
(Don't evaluate the integral, just determine a good choice of parts.)

Example: A Sneaky Choice of Parts

$$\int \ln(x)\ dx$$

Activity: Which Method? (3 min)

Consider the following indefinite integrals. Which method would you try first?
(You don't need to evaluate the indefinite integral. Just pick a tool and say how you'd apply it.)

1. $\int x^2 + 2x \; dx$
2. $\int xe^{-x^2} \; dx$
3. $\int x^2 + \cos(2x) \; dx$
4. $\int x^2 \cos(x) \; dx$
5. $\int e^x \cos(x) \; dx$

Currently our methods are: direct, substitution, or parts.

Week 11: Partial Fractions

Remark: Division of Polynomials

This week, we're going to explore a technique called "Partial Fractions". This is not so much a technique of integration, but a means of simplifying integrands so that they become simpler. It builds on the notion of long division. We first generalize the notion of division with remainder of integers to polynomials.

Theorem: Integer Long Division

For any rational number, we can write:

$$\frac{p}{q} = n + \frac{r}{q}$$

where $0 \leq r < q$.

Example: Long Division with Remainder

Find the remainder when 1221 is divided by four. Write your answer in the format:

$$\frac{1221}{4} = n + \frac{r}{4}$$

where $0 \leq r < 4$.

Example: Long Division with Remainder

Find the remainder when 123 is divided by 11. Write your answer in the format:

$$\frac{123}{11} = n + \frac{r}{11}$$

where $0 \leq r < 11$.

Remark: The Big Leap to Polynomials

We now make the big leap to polynomial long division. The idea here is that we treat each degree as a "digit". We then proceed to handle the division "digit" by "digit".

Example: Polynomial Division

Divide the polynomial $x^2 + 2x + 1$ by $x + 1$ and express you answer in the format:

$$\frac{x^2 + 2x + 1}{x + 1} = p(x) + \frac{r(x)}{x + 1}$$

where $0 \leq \deg(r) < \deg(x + 1)$.

Example: Polynomial Division

Divide the polynomial $x^3 - 1$ by $x + 3$ and express you answer in the format:

$$\frac{x^3 - 1}{x + 3} = p(x) + \frac{r(x)}{x + 3}$$

where $0 \leq \deg(r) < \deg(x + 3)$.

Theorem: Polynomial Long Division

For any rational function, we can write:

$$\frac{f(x)}{q(x)} = p(x) + \frac{r(x)}{q(x)}$$

where $\deg(r) < \deg(q)$. (This lets us simplify the integrand by reducing the degree.)

Example: A Rational Function

$$\int \frac{x^3 - 1}{x + 3} dx$$

Note: We've already calculated this polynomial long division.

Example: A Rational Function

$$\int \frac{x^2 + 2x + 1}{x + 3} dx$$

Remark: Partial Fractions

We now get to the mechanics of actual partial fractions decomposition. If a rational function $p(x)/q(x)$ has $\deg(p) < \deg(q)$ then we might be able to re-write it in a simpler format. We give a few examples before beginning the general theory. We also illustrate two distinct methods of finding the unknown coefficients.

Example: A Surprising Equality

Use the equality

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$$

to compute the integral

$$\int \frac{1}{1-x^2} dx.$$

Example: Re-write The Polynomial as A Sum

Write the rational function in the given format: $\dfrac{1}{x^2+5x+6} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$.
Method #1: Find the values A and B by setting up a linear system with two equations and two unknowns.

Example: Re-write The Polynomial as A Sum

Write the rational function in the given format: $\dfrac{1}{x^2 - 3x + 2} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$.
Method #2: Find the values of A and B by subbing in specific values of x.

Activity: Try it! (5 min)

Write the rational function in the given format: $\dfrac{2x-1}{(x-3)(x+5)} = \dfrac{A}{x-3} + \dfrac{B}{x+5}$.
How should we adapt Method #1 and #2 when there is a non-constant numerator?

Example: Three Roots

Produce a partial fractions decomposition of

$$\frac{6+4x}{6+11x+6x^2+x^3}$$

given the factorization $6+11x+6x^2+x^3 = (x+1)(x+2)(x+3)$.

Activity: What Goes Wrong? (2 min)

Try to write the rational function in the given format: $\dfrac{x+4}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x+1}$.
What goes wrong? What is weird here?

Example: A Decomposition with a Repeated Factor

Write the rational function in the given format: $\dfrac{x+4}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$.

Activity: Try it! (5 min)

Write the rational function in the given format: $\dfrac{2x+3}{(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$.

Week 12: Area and Volume

Remark: Riemann Sums and Area

Recall that we developed the theory of Riemann sums to measure area.

$$\text{“The signed area bounded by } y = f(x) \text{ on } [a, b]\text{”} = \lim_{N\to\infty} \sum_{k=0}^{N} f(x_k^*)\Delta x_k$$

This week, we'll look at applying this concept to measure area and volumes.

Example: What Does Riemann Really Measure?

Sketch the regions bounded by $y = x$, $x = -1$, $x = 1$, and $y = 0$.
Use a definite integral to measure this “area”. Why is the answer surprising?

Theorem: Area Between Curves

If $g(x) \leq f(x)$ on the interval $[a, b]$ then the area bounded by $x = a$, $x = b$, $y = f(x)$ and $y = g(x)$ is:

$$A = \int_a^b f(x) - g(x) dx$$

Example: The Unsigned Area

Use highschool geometry to measure the true area of the region in the previous example.
Setup and evaluate an integral to measure its area.

Example: Complicated Integrals / Top and Bottom Curves

Find the area bouned by $y = xe^x$, $y = e^x$, $x = 0$, and $x = 1$.

Remark: Curves Which Cross

It is inconvenient that we need $g(x) \leq f(x)$ over the whole interval $[a, b]$ because curves tend to cross each other. We can get rid of this restriction by considering:

$$A = \int_a^b |f(x) - g(x)| \, dx$$

Example: A Pair of Curves Which Cross

Find the unsigned area bounded by $y = x$ and $y = x^3$.

Example: More Curves Which Cross

Find the unsigned area bounded by $y = \cos(x)$ and $y = \frac{1}{2}$ over $0 \leq x \leq \pi$.

Example: Complicated Regions

Sketch the region bounded by $y = x$, $y = 0$, and $y = (x - 1)^2$. Find the area of this region.

Activity: Try it! (5 min)

Use Desmos to sketch the region bounded by: $y = 0$, $y = 1 - |x|$, $y = x^2$.
Setup and evaluate an integral to compute this area.

Remark: Area with Respect to y.

An notable property of area is that is doesn't "care about" how the region is bounded. It doesn't "care about" $y = f(x)$ versus $x = g(y)$. The region is bounded and has an area independent of the functions.

Example: An Area With Respect to y.

Find the area bounded by $x = y^2$ and $x = 9$.

🏃 Activity: Lukas' Problem (5 min)

Find the line $y = c$ that cuts the region bounded by $y = x^2$ and $y = 1$ in half.

Remark: Area and Volume

We've seen that we can get area as an integral.

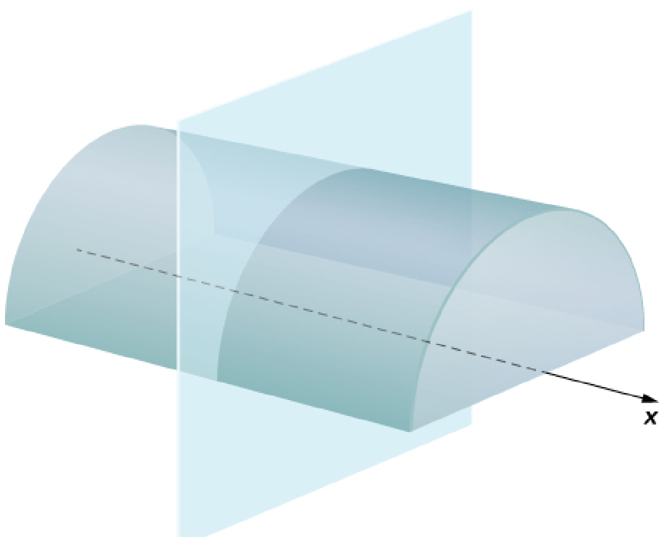
$$\text{“Area”} = \int_a^b \text{“Length”}\, dx$$

If we go one dimension higher, we can get:

$$\text{“Volume”} = \int_a^b \text{“Area”}\, dx$$

This is sometimes called Cavalieri's Principle after Galileo's student Bonaventura Cavalieri (1598-1647).

- Examine the solid and determine the shape of a cross-section of the solid.
  It is often helpful to draw a picture if one is not provided.
- Determine a formula for the area of the cross-section.
- Integrate the area formula over the appropriate interval to get the volume.



Three-dimensional cylinder

Two-dimensional cross section

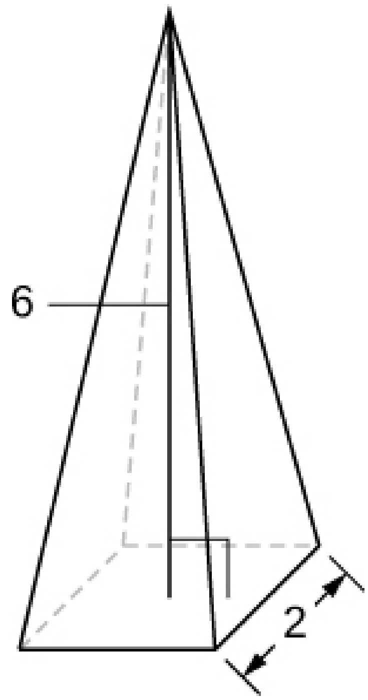
OpenStax §6.2 Determining Volume by Slicing

Example: Find The Volume of a Sphere

Find the volume of a sphere of radius $R > 0$ by slicing it parallel to the xy-plane and integrating $\text{Area}(z)$.

Example: A Pyramid **(OS §6.2 Q63)**

Find the volume of a pyramid of height six units and square base of sidelength two units.



Definition: Volume of Revolution

We obtain a **solid of revolution** by revolving a curve $y = f(x)$ over $[a, b]$ around the x-axis.
The volume of such a solid is a **volume of revolution**. (See the Desmos example below.)

Example: Find the Volume of a Sphere

Find the volume of the sphere of radius $R > 0$ as a volume of revolution.

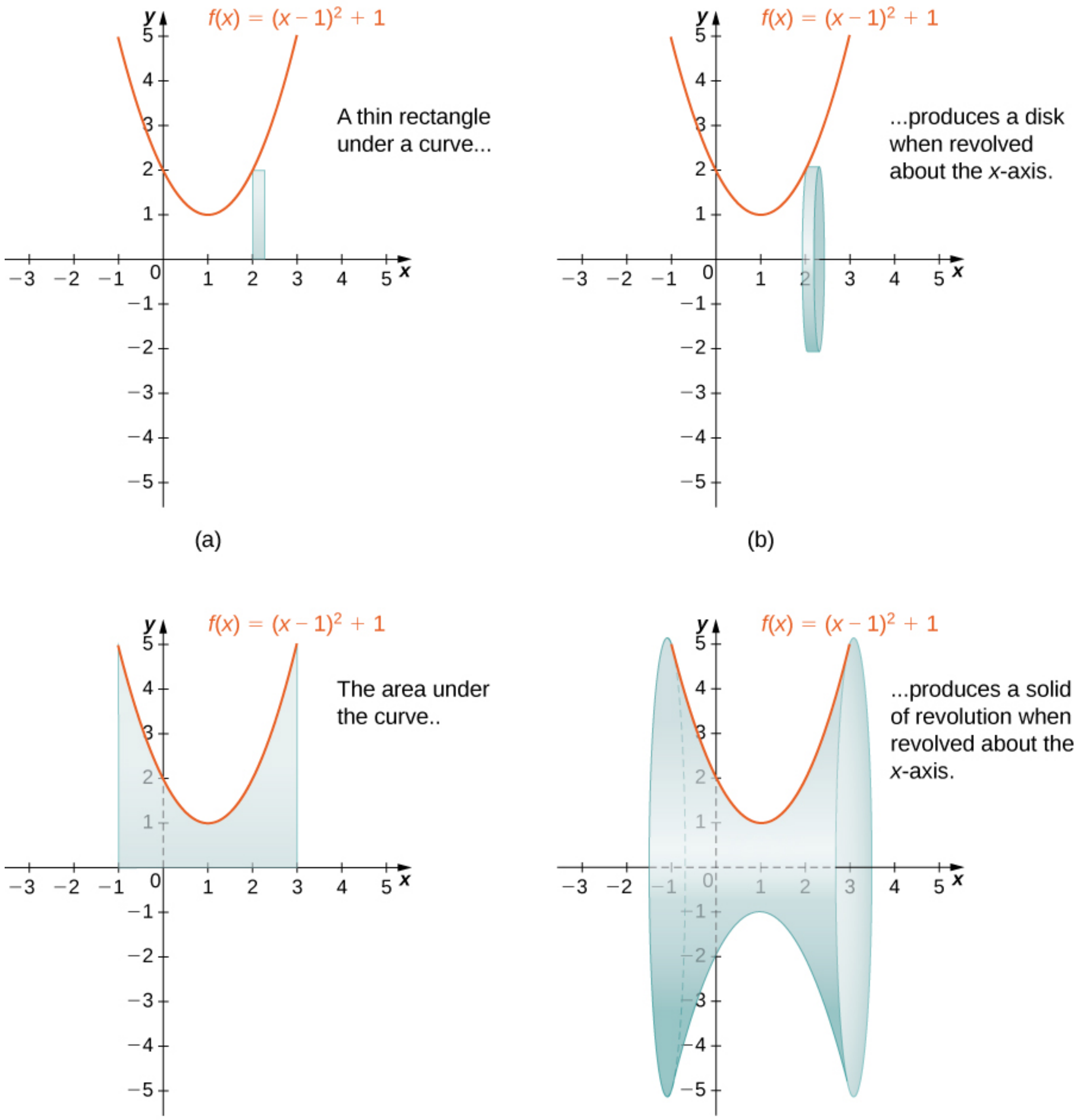
https://www.desmos.com/3d/1a7f86ac77

Theorem: The Disk Method

If $f(x)$ is non-negative on $[a,b]$ then the volume of revolution generated by $y = f(x)$ over $[a,b]$ is:

$$V = \int_a^b \text{Area}(x)\ dx = \int_a^b \pi[f(x)]^2 dx$$

OpenStax §6.2 Determining Volume by Slicing



(a) (b) (c) (d)

Example: Applying the Disk Method

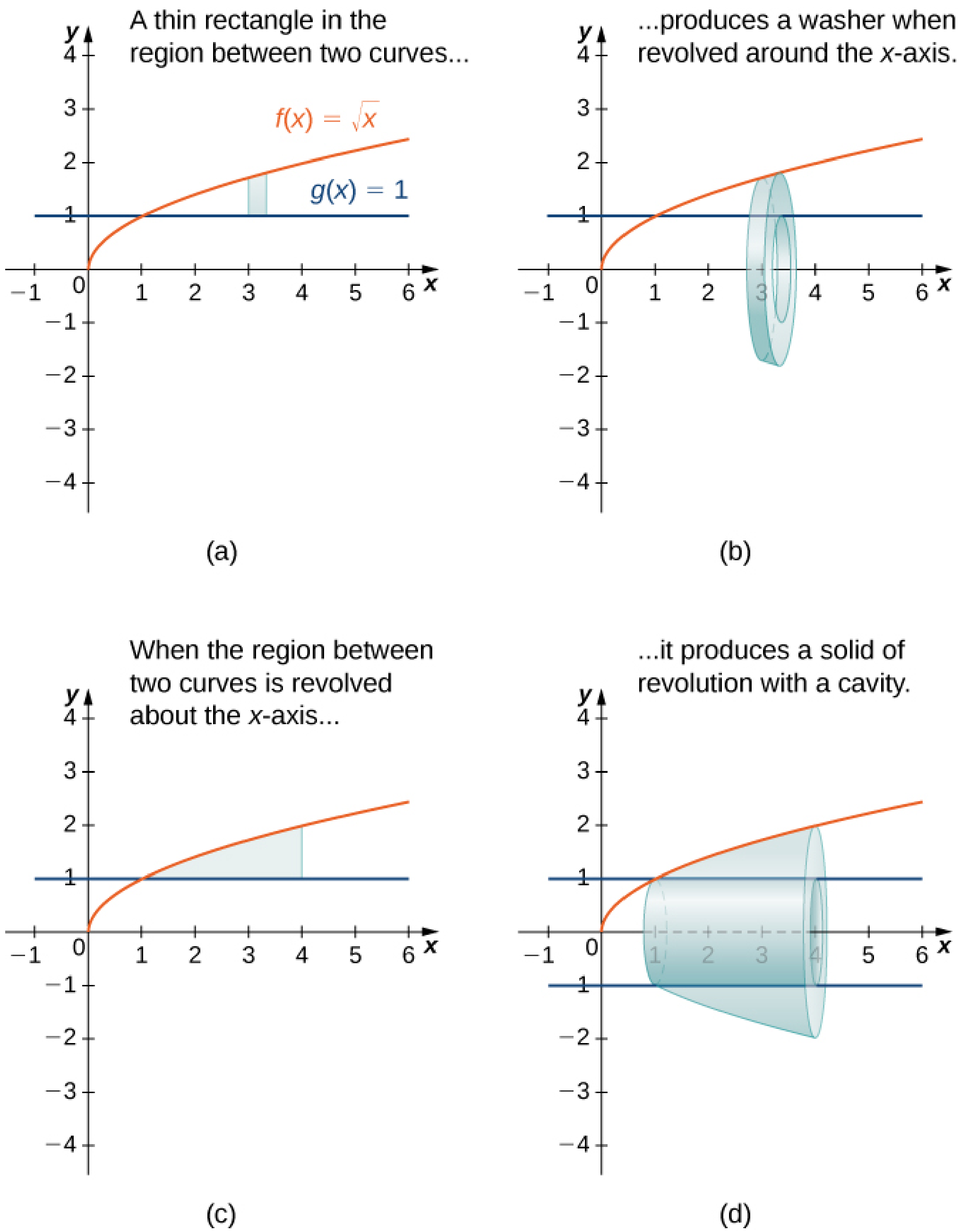
Find the volume of revolution of $y = (x-1)^2 + 1$ on the interval $[-1, 3]$.
(This is the example illustrated by the previous graphic.)

Theorem: The Washer Method

If $g(x) \leq f(x)$ are both non-negative on $[a, b]$ then the volume of revolution generated the region bounded by them over $[a, b]$ is:

$$V = \int_a^b \text{Area}(x)\ dx = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2\right) dx$$

OpenStax §6.2 Determining Volume by Slicing



Example: Applying the Washer Method

Find the volume of revolution of $y = \sqrt{x}$ and $y = 1$ on the interval $[1, 4]$.
(This is the example illustrated by the previous graphic.)